Ordinary Differential Equation

(1) Solve the following differential equations: (30%)

(a) $2y(t)y''(t) + y'^2(t) = 0$.

(b) $ty'(t) - y(t) = t^\gamma, \quad \gamma \geq 1$.

(c) $y''(t) - 4y(t) = f(t)$.

(d) $dz = (x^2 + 2xy - y^2)dx + (x^2 - 2xy - y^2)dy$.

(2) (Picard's iteration) Consider the differential equation (20%)

$$y'(t) = f(y(t), t), \quad f \in C(\mathbf{R} \times \mathbf{R}^+),$$

determined by a function y in a domain of the extended phase space $\mathbf{R}^2$. Then the *Picard mapping* is defined by

$$(Ay)(t) = y_0 + \int_0^t f(y(\tau), \tau)d\tau, \quad y_0 = y(0),$$

and the *succesive Picard approximations* is given by

$$\phi_0 = y_0, \phi_1 = A\phi_0, \cdots \phi_n = A\phi_{n-1} = A^n\phi_0, \quad \forall n = 1, 2, 3, \cdots$$

(a) Use the Picard's iteration to construct a sequence of approximate solutions of the following differential equation

$$y'(t) = k(y + 2), \quad y(0) = 1, \quad k \in \mathbf{R},$$

and prove that the limit of the approximate solution is the solution. (You need to give a rigorous proof!)

(b) Discuss the long time behaviour, i.e., $t \to \infty$, of the solution.

(c) Discuss the behaviour of the solution as $k \to 0$.

(3) Consider the linear system (20%)

$$\frac{dx}{dt} = \lambda x + y\,, \quad \frac{dy}{dt} = \mu y\,, \quad (x(0), y(0))^t = (x_0, y_0)^t$$

where λ and μ are real constants.

(a) Find out and classify the critical point then determine whether it is stable or unstable.

(b) Sketch the trajectories in the phase plane.

(c) Solve the linear system.

(4) Show that there is at most one solution of the initial value problem (10%)

$$y'' + e^y = 1\,, \quad \text{for } t > 0\,,$$
$$y(0) = 1\,, \quad y'(0) = 0\,.$$

(5) Let S_t be the sphere $(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2 = t^2$ and $f(\xi, \eta, \zeta)$ continuous. Prove rigorously that the function given by

$$u(x, y, z, t) = \frac{1}{4\pi} \iint_{S_t} \frac{f(\xi, \eta, \zeta)}{t} dS\,,$$

satisfies the initial value problem of the wave equation (20%)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial t^2}$$
$$u(x, y, z, 0) = 0\,,$$
$$\frac{\partial}{\partial t} u(x, y, z, 0) = f(x, y, z)\,.$$