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1. Let $\{x_n\}$ be an arbitrary real sequence. Judge whether or not the following statements are correct and briefly explain your answers.
   (a) Suppose for all $\varepsilon > 0$, there exists a positive integer N such that whenever $n > N$ implies $x_n < \varepsilon$, then $\lim_{n\to\infty} x_n = 0$. (4%)
   (b) $\lim_{n\to\infty} x_n = a \iff \lim_{n\to\infty} |x_n| = |a|$. (6%)

2. Assume $f''(x) = e^{\frac{x-1}{2}}(x+4), f'(1) = 10$. Please find
   (a) where f is concave up and concave down? (4%)
   (b) where f is increasing and decreasing? (6%)

3. (a) State and prove the Fundamental Theorem of Calculus. (8%)
   (b) Find $\lim_{x\to 0+} \dfrac{\int_1^{-\ln x} \frac{1}{t}\,dt}{\cot x}$. (7%)

4. Let $f(x) = \begin{cases} 2 - x^2(2 + \sin\frac{1}{x}), & x \neq 0 \\ 2, & x = 0. \end{cases}$
   (a) Show that $x = 0$ is a global maximum point of $f(x)$ on R. (3%)
   (b) Compute $f'(x)$ for $x = \frac{1}{k\pi}$, $k \in Z\backslash\{0\}$. (5%)
   (c) From (b), conclude that, $\forall\varepsilon > 0$, $f(x)$ is NOT monotonically decreasing on $(0, \varepsilon]$, nor is it monotonically increasing on $[-\varepsilon, 0)$. (7%)

5. (a) Show that $\int_1^\infty \frac{1}{x^p}\,dx$ diverges for $p \le 1$ and converges for $p > 1$. (5%)
   (b) Test convergence(divergence) for $\int_2^\infty \frac{1}{x\ln x}\,dx$ and $\sum_{n=2}^\infty \frac{1}{n[\ln n]^2}$. (8%)
   (c) Give an example of a region in the first quadrant that gives a solid of finite volumn when revolved about the $x-axis$ but gives a solid of infinite volumn when revolved about $y-axis$. (7%)

6. (a) For $f(x) = e^x + e^{-x} + 2\cos x$, compute the third order Taylor's expansion around $x = 0$ with remainder R_4. (7%)
   (b) Use (a) to show $f(x)$ has a local minimum at $x = 0$. (7%)
   (c) Explain why $f(t) = \begin{cases} 0, & t < 0 \\ t^4, & t \ge 0 \end{cases}$ can not be represented by a *Taylor* series. (6%)

7. Let

$$S(x) = \frac{x+2}{3} + \frac{(x+2)^2\ln 2}{2\cdot 9} + \frac{(x+2)^3\ln 3}{3\cdot 27} + \frac{(x+2)^4\ln 4}{4\cdot 81} + \cdots$$

be a power series with its n^{th} term following the same pattern as the first 4 terms.
   (a) Determine the convergence set for $S(x)$. (5%)
   (b) Are there any relations between the convergence set of $S'(x)$ and $S(x)$? (5%)