Differential Equations

(15 points each for the first two problems, 14 points each for the others)

1. For the undamped pendulum equation
$$L\theta'' + g\sin\theta = 0,$$
draw the phase plane and locate the attracting points. Note that a phase plane is a plot of the derivative of the dependent variable against the dependent variable in a second-order equation.

2. For the linear system of equations
$$\frac{dx}{dt} = 2x - y - 2, \quad \frac{dy}{dt} = 3x - 2y - 2,$$
find all critical points and investigate stability (asymptotically stable, stable, or unstable) of each.

3. Find the eigenvalues and eigenfunctions of the eigenvalue problem
$$X''(x) + \lambda X(x) = 0, \quad X'(0) = X(\pi) = 0.$$

4. Calculate by hand the approximating values of $x(0.2)$ and $y(0.2)$ of the initial value problem
$$x' = 9x + 5y, \quad x(0) = 1,$$
$$y' = 3x + 2y, \quad y(0) = 1,$$
by using the Euler method with two steps of size $h = 0.1$.

5. Find the Laplace transform of the solution of the problem
$$x'' + 4x = \sin 2t, \quad x(0) = 1,\ x'(0) = 4.$$

6. Find a second solution of
$$x^2y'' - 3xy' + 3y = 0$$
in the form $y_2 = u(x)y_1$, given the first solution $y_1 = x$.

7. For the differential equation
$$x^2y'' + xy' + (x^2 - \tfrac{1}{16})y = 0,$$
find the indicial equation and its roots at each regular singular point.