成功大學 98 學年度碩士班甄試入學考試(基礎數學)試題

Pert I.

Advanced Calculus

1. Suppose that $\lim_{x\to\infty} x \cdot f(x) = L$. Prove that $\lim_{x\to\infty} f(x) = 0$. (10 points)

2. Define a function f by

$$f(x) = \begin{cases} x \cdot \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Prove or disprove that f is uniformly continuous on $\mathbb{R}$. (10 points)

3. Prove that

$$\left|\int_0^1 x \cdot \sin(1/x)\,dx\right| \leq \left(\int_0^1 x^2 \cdot \sin^2(1/x)\,dx\right)^{1/2}.$$

(Hint:$(\sum a_k b_k)^2 \leq \sum a_k^2 \sum b_k^2$.) (10 points)

4. Let f be a smooth function on $(-1,1)$ and $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $x \in (-1,1)$. Let $\{x_m\}$ be a sequence with $x_m \neq 0$ for all $m \in \mathbb{N}$. Assume that $\{x_m\}$ converges to zero with $f(x_m) = 0$. Show that $f = 0$ on $(-1,1)$. (10 points)

5. For $x \in \mathbb{R}^3$, let $\rho(x)$ be a charge density that is continuous and such that $\rho(x) = 0$ for $\|x\|_2 > 1$. Show that the electrostatic potential, given by

$$\phi(x) = \frac{1}{4\pi} \iiint_{\mathbb{R}^3} \rho(y)/\|x - y\|_2\,dy,$$

is a convergent integral for each $x \in \mathbb{R}^3$. (10 points)

Part II. **Linear Algebra (2008)**

Please answer all questions and show all your works (50 points)

We use the following notations:
$\mathbb{R}$: the set of all real numbers,
$\mathbb{C}$: the set of complex numbers,
A^T : the transpose matrix of the $n \times n$ matrix A,
I_n : the $n \times n$ identity matrix,
$\det A$: the determinant of the $n \times n$ matrix A.

1. (15 points) An $n \times n$ matrix A with entries in $\mathbb{R}$ is said to be an orthogonal matrix if $AA^T = I_n$.
   (a) (3 points) Assume that A is an $n \times n$ orthogonal matrix with entries in $\mathbb{R}$. Prove that $\det A = 1$ or $\det A = -1$.
   (b) (4 points) Assume that A is a 2×2 orthogonal matrix with entries in $\mathbb{R}$. Prove that there is an orthogonal 2×2 matrix U such that
   $$U^T AU = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{or} \quad U^T AU = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \text{ for some } \alpha \in [0, 2\pi].$$
   (c) (8 points) Assume that A is a 3×3 orthogonal matrix with entries in $\mathbb{R}$ such that $\det A = 1$. Prove that there is an orthogonal 3×3 matrix U such that
   $$U^T AU = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{or} \quad U^T AU = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \text{ for some } \alpha \in [0, 2\pi].$$

2. (15 points) Let A be an $n \times n$ matrix with entries in $\mathbb{R}$ and let $\chi(x)$ be the characteristic polynomial of A. Assume that there is a $\lambda \in \mathbb{R}$ such that $\chi(\lambda) = 0$.
   (a) (5 points) Prove that there is a nonzero $n \times 1$ matrix with entries in $\mathbb{R}$ such that $Av = \lambda v$.
   (b) (10 points) Assume that $\chi(x) = (x - \lambda)^2 f(x)$ for some polynomial $f(x)$ with coefficient in $\mathbb{R}$. Prove that either the dimension of the kernel of $A - \lambda I$ is greater 2 or there are two nonzero $n \times 1$ matrices v_1 and v_2 with entries in $\mathbb{R}$ such that $Av_1 = \lambda v_1$ and $Av_2 = \lambda v_2 + v_1$.

3. (10 points) Let V be the set of all 2×2 matrices with entries in $\mathbb{C}$. Let f denote the linear transformation from V to V defined by $f(A) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} A - A \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ for all $A \in V$. Find a Jordan canonical form of f and an ordered basis for V so that the matrix associated to f with respect to the ordered basis is the Jordan canonical form.

4. (10 points) Let $\{e_1, e_2, e_3\}$ be a basis for the a 3-dimensional vector space V over $\mathbb{R}$. Let g be a linear transformation from V to V defined by $g(\sum_{i=1}^{3} a_i e_i) = (-5a_2 + a_3)e_1 + a_2 e_2 + (2a_1 + 2a_2)e_3$ for all $a_1, a_2, a_3 \in \mathbb{R}$. A subspace W of V is said to be an invariant subspace of g if $g(W) \subseteq W$.
   (a) (2 points) Find the matrix representation of g with respect to the basis $\{e_1, e_2, e_3\}$.
   (b) (5 points) Find all invariant subspaces of g.
   (c) (3 points) For each invariant subspace W, find a basis for the quotient vector space V/W.