Part I. LINEAR ALGEBRA

1. (8 points) Let A be a 3×4 matrix over the real number field $\mathbb{R}$, and let $\{(2,3,1,0)\}$ be a basis for the nullspace of A.
   (a) What is the rank of A and the complete solution to $A\mathbf{x}=\mathbf{0}$?

   Solution: The dimension of the nullspace is 1, so the rank of A is $4-1=3$. The complete solution to $Ax=0$ is $x=c\cdot(2,3,1,0)$ for any constant c.

   (b) Find a basis for the column space of A^T.

   Solution: The column space of A^T is the row space of A. The nonzero rows of the row reduced echelon form $\begin{bmatrix}1&0&-2&0\\0&1&-3&0\\0&0&0&1\end{bmatrix}$ form a basis.

2. (a) (4 points) The linear transformation $T:\mathbb{R}^2\to\mathbb{R}^2$ reflects a vector about the line $y=-x$ and then projects that vector orthogonally onto the x-axis. Find the standard matrix for T.

   Solution: $T\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ and $T\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}-1\\0\end{bmatrix}$ so the matrix representation for T is $\begin{bmatrix}0&-1\\0&0\end{bmatrix}$.

   (b) (4 points) Suppose $T:\mathbb{R}^4\to\mathbb{R}^2$ is a linear transformation with $T(1,0,0,1)=(2,3)$ and $T(0,1,1,0)=(1,5)$. Find $T(6,2,2,6)$.

   Solution: Let $v_1=(1,0,0,1)$ and $v_2=(0,1,1,0)$. Then $v=(6,2,2,6)=6v_1+2v_2$. By linearity,

$$T(v)=T(6v_1+2v_2)=6T(v_1)+2T(v_2)=6(2,3)+2(1,5)=(14,28)$$

3. (8 points) Suppose the 3×3 matrix A over $\mathbb{R}$ has eigenvalues 0, 1, and 2 with eigenvectors v_0, v_1, v_2, respectively.
   (a) What is the trace of $A-2I$?

   Solution: $A-2I$ has eigenvalues $-2,-1,0$ so its trace is -3.

   (b) Solve the equation $Ax=v_1+v_2$ for x.

   Solution: $x=av_0+v_1+\frac{1}{2}v_2$.

4. (16 points) Let V be the vector space $\mathbb{R}^3$ over $\mathbb{R}$. The following matrix is a *projection matrix* on V: $P=\frac{1}{21}\begin{bmatrix}1&2&-4\\2&4&-8\\-4&-8&16\end{bmatrix}$.
   (a) What subspace W of V does P project onto?

   Solution: The projection matrix P projects onto the column space of P which is the line $c\cdot(1,2,-4)$.

   (b) What is the distance from that subspace W to $\mathbf{b}=(5,4,-2)$?

   Solution: The vector from $\mathbf{b}$ to the subspace is

$$\mathbf{e}=\mathbf{b}-P\mathbf{b}=\begin{bmatrix}5\\4\\-2\end{bmatrix}-\frac{21}{21}\begin{bmatrix}1\\2\\-4\end{bmatrix}=\begin{bmatrix}4\\2\\2\end{bmatrix}$$

   and the distance is

$$\|\mathbf{e}\|=\sqrt{4^2+2^2+2^2}=2\sqrt{6}.$$

   (c) What are the three eigenvalues of P?

   Solution: Since P projects onto a line, its three eigenvalues are $0,0,1$. The eigenvectors for 0 are vectors orthogonal to $(1,2,-4)$. The eigenvector for 1 is $(1,2,-4)$.

   (d) If you solve $\frac{du}{dt}=-Pu$ (notice minus sign) starting from $u(0)$, the solution $u(t)$ approaches a steady state as $t\to\infty$. Describe that limit vector $u(\infty)$?

Solution: The solution $u(t)$ to the differential equation has the form $u(t) = v_1 e^{-t} + v_2$ where v_1 is in W and v_2 is in the orthogonal complement of W. Then $u(\infty) = v_2$, which is the projection of $u(0)$ onto the orthogonal complement of W. That is, $u(\infty) = u(0) - P(u(0))$.

5. (10 points) Let n denote a positive integer, V denote an n-dimensional vector space, and T denote a linear operator on V. Suppose $v \in V$ is a nonzero vector such that $T^k v = 0$ for some positive integer k. Show that $T^n v = 0$.

Solution: Suppose k is the smallest positive integer such that $T^k v = 0$. The vectors $v, Tv, T^2 v, \ldots, T^{k-1} v$ are linearly independent so $k \le n$:
Suppose $c_0 v + c_1 Tv + c_2 T^2 v + \cdots + c_{k-1} T^{k-1} v = 0$. Applying T^{k-1} to both sides we get $c_0 T^{k-1} v = 0$ and so $c_0 = 0$. Now applying T^{k-2} to both sides we get $c_1 T^{k-1} v = 0$ and so $c_1 = 0$. Continuing in this fashion, we see that $c_j = 0$ for all j.

Part II. ADVANCED CALCULUS

6. (10 points) Let $x_1 = \frac{\pi}{2}$ and suppose that x_n are defined inductively for $n = 2, 3, \ldots$ by $\cos(x_{n+1}) = \frac{\sin(x_n)}{x_n}$, $0 < x_{n+1} < \frac{\pi}{2}$, for $n = 1, 2, \ldots$.

   (a) Prove that $x_{n+1} < x_n$ for $n = 1, 2, \ldots$. Hint: You may find $\cos(x) = \frac{d}{dx}\sin(x)$ and the Mean Value Theorem useful.

   Solution: By the Mean Value Theorem, there exists a $0 < c < x_n$ such that $\sin x_n = \sin x_n - \sin 0 = \cos c\,(x_n - 0) = \cos c\, x_n$. This implies that $x_{n+1} = c < x_n$.

   (b) Show that the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent.

   Solution: Since the set $\{x_n | n \in \mathbb{N}\}$ is nonempty and bounded from below by 0, $l = \inf\{x_n | n \in \mathbb{N}\}$ exists. The definition of l implies that for each $\varepsilon > 0$ there exists an x_k for some $k \in \mathbb{N}$ such that $l + \varepsilon > x_k$. The result of part (a) says that $\{x_n\}_{n=1}^{\infty}$ is decreasing which implies that $l + \varepsilon > x_k \ge x_n > 0$ for all $n \ge k$. Therefore, we get $|x_n - l| < \varepsilon$ for all $n \ge k$ which means that $\lim_{n\to\infty} x_n = l$.

   (c) Find explicitly the limit x of the sequence $\{x_n\}_{n=1}^{\infty}$.

   Solution: Let $f(x) = x\cos x - \sin x$ for $x \in [0, \frac{\pi}{2}]$. To find all possible limits is equivalent to find zeros of f over $[0, \frac{\pi}{2}]$. Note that $f(0) = 0$ and $f'(x) = -x\sin x < 0$ for all $x \in (0, \frac{\pi}{2}]$, we can conclude that $x = 0$ is the only zero of f. Hence, $\lim_{n\to\infty} x_n = 0$.

7. (a) (5 points) State what it means for a sequence $\{f_n(x)\}_{n=1}^{\infty}$ of real valued functions on a set $X \subset \mathbb{R}^p$ to converge uniformly to a function f on X.

   Solution: $\{f_n\}_{n=1}^{\infty}$is said to converge uniformly on X to f if for each $\varepsilon > 0$ there exists a $K(\varepsilon) \in \mathbb{N}$ such that for all $n \ge K(\varepsilon)$ and $x \in X$ then $|f_n(x) - f(x)| < \varepsilon$.

   (b) (5 points) Let $F : [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function on the unit square. Let $f_n(t) = F(\frac{1}{n}, t)$ and $f(t) = F(0,t)$. Use the definition in part (a) to show that $\{f_n(x)\}_{n=1}^{\infty}$ converges uniformly to f on the interval $[0,1]$.

   Solution: Since F is continuous on $[0,1] \times [0,1]$, F is uniformly continuous there. This implies that for any $\varepsilon > 0$ and any $t \in [0,1]$, there exists a $\delta = \delta(\varepsilon) > 0$ such that if $|x - 0| < \delta$ then $|F(x,t) - F(0,t)| < \varepsilon$. Letting $K(\varepsilon) = [\frac{1}{\delta}] + 1$, where $[\frac{1}{\delta}]$ is defined to be the greatest integer less than or equal to $\frac{1}{\delta}$, we note that $n \ge K(\varepsilon) > \frac{1}{\delta}$ implies that $\frac{1}{n} < \delta$. Thus, $|f_n(t) - f(t)| = |F(\frac{1}{n}, t) - F(0,t)| < \varepsilon$ for all $n \ge K(\varepsilon)$. Thus $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on $[0,1]$

8. (10 points) For each integer $k \ge 1$, let $f_k : \mathbb{R} \to \mathbb{R}$ be differentiable satisfying $|f_k'(x)| \le 1$, for all $x \in \mathbb{R}$, and $f_k(0) = 0$.

   (a) For each $x \in \mathbb{R}$, prove that the set $\{f_k(x)\}_{k=1}^{\infty}$ is bounded.

   Solution: For each $x \in \mathbb{R}$, since $|f_k(x)| = |f_k(x) - f_k(0)| \le |f_k'(c_k)\,(x - 0)| \le |x|$, where c_k lies between x and 0, the set $\{f_k(x)\}_{k=1}^{\infty}$ is bounded.

   (b) Use Cantor's diagonal method to show that there is an increasing sequence $n_1 < n_2 < n_3 < \cdots$ of positive integers such that, for every $x \in \mathbb{Q}$, we have $\{f_{n_k}(x)\}$ is a convergent sequence of real numbers.

Solution: Let $\mathbb{Q} = \{x_1, x_2, \cdots\}$. Since $\{f_k(x_1)\}$ is bounded, we can extract a convergent subsequence, denoted $\{f_k^1(x_1)\}$, out of $\{f_k(x_1)\}$. Next, the boundedness of $\{f_k^1(x_2)\}$ implies that we can extract a convergent subsequence, denoted $\{f_k^2(x_2)\}$, out of $\{f_k^1(x_2)\}$. Continuing this way, the boundedness of $\{f_k^j(x_{j+1})\}$ implies that we can extract a convergent subsequence $\{f_k^{j+1}(x_{j+1})\}$, out of $\{f_k^j(x_{j+1})\}$ for each $j \geq 1$. Let $f_{n_k} = f_k^k$ for each $k \geq 1$. Then $\{f_{n_k}\}$ is a subsequence of $\{f_n\}$ and $\{f_{n_k}\}$ converges at each $x_j \in \mathbb{Q}$.

9. (10 points) A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *convex* if for all $x, y \in \mathbb{R}$, $\lambda \in [0,1]$, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is convex and that $f''(x)$ exists for all $x \in \mathbb{R}$.

(a) For any $x < y$ and $0 < h < y - x$, prove that

$$f(x+h) \leq \frac{y-x-h}{y-x} \cdot f(x) + \frac{h}{y-x} \cdot f(y), \quad \text{and} \quad f(y-h) \leq \frac{h}{y-x} \cdot f(x) + \frac{y-x-h}{y-x} \cdot f(y).$$

Solution: Setting $x + h = \lambda x + (1-\lambda)y$, we get $\lambda = \frac{y-x-h}{y-x}$ and $1 - \lambda = \frac{h}{y-x}$. Thus

$$f(x+h) = f(\tfrac{y-x-h}{y-x}x + \tfrac{h}{y-x}y) \leq \tfrac{y-x-h}{y-x} f(x) + \tfrac{h}{y-x} f(y).$$

Setting $y - h = \beta x + (1-\beta)y$, we get $\beta = \frac{h}{y-x}$ and $1 - \beta = \frac{y-x-h}{y-x}$. Thus

$$f(y-h) = f(\tfrac{h}{y-x}x + \tfrac{y-x-h}{y-x}y) \leq \tfrac{h}{y-x} f(x) + \tfrac{y-x-h}{y-x} f(y).$$

(b) Prove that $f'(x) \leq f'(y)$ whenever $x \leq y$.

Solution: Since

$$f'(x) - f'(y) = \lim_{h \to 0^+} \tfrac{f(x+h)-f(x)}{h} - \tfrac{f(y-h)-f(y)}{-h} = \lim_{h \to 0^+} \tfrac{f(x+h)+f(y-h)-f(x)-f(y)}{h} \leq \lim_{h \to 0^+} \tfrac{f(x)+f(y)-f(x)-f(y)}{h} = 0,$$

where we have used the result of part (a) in the last inequality. We have shown that $f'(x) \leq f'(y)$ whenever $x \leq y$.

(c) Prove that $f''(x) \geq 0$ for all $x \in \mathbb{R}$.

Solution: The result of part (b) implies that $f''(x) = \lim\limits_{h \to 0^+} \frac{f'(x+h)-f'(x)}{h} \geq 0$, we have $f''(x) \geq 0$ for all $x \in \mathbb{R}$.

10. (10 points) Let $g : \mathbb{R}^p \to \mathbb{R}^p$ belong to class $C^1(\mathbb{R}^p)$, i.e. $Dg(x)$ exists for all $x \in \mathbb{R}^p$ and the mapping $x \to Dg(x)$ is continuous. Assume that there is a constant a such that $\|Dg(x)\| \leq a < 1$ for each $x \in \mathbb{R}^p$.

(a) Show that the function $f(x) = x + g(x)$ for $x \in \mathbb{R}^p$ satisfies $\|f(x_1) - f(x_2) - (x_1 - x_2)\| \leq a\|x_1 - x_2\|$ for all $x_1, x_2 \in \mathbb{R}^p$.

Solution: The Mean Value Theorem implies that there exists a $z = \lambda x_1 + (1-\lambda)x_2 \in \mathbb{R}^p$ for some $\lambda \in [0,1]$ such that

$$\|f(x_1) - f(x_2) - (x_1 - x_2)\| = \|Dg(z) \cdot (x_1 - x_2)\| \leq a\|x_1 - x_2\|.$$

(b) Show that f in part (a) is a bijection of $\mathbb{R}^p$ into $\mathbb{R}^p$.

Solution: Since $Df = I + Dg$ and the eigenvalues of Dg are bounded by $a < 1$, Df is invertible everywhere. The Inverse Function Theorem implies that f is a local bijection. Since the result of part (a) says that f is a global one-to-one function of $\mathbb{R}^p$ into $\mathbb{R}^p$, f is a bijection of $\mathbb{R}^p$ into $\mathbb{R}^p$.