92 academic year

Part I.

I.1. Let $f_n : I \to \mathbb{R}$, $n \geq 1$, I is an interval in $\mathbb{R}$ and f_n converges to f uniformly. Prove or disprove

(a) if f_n is continuous, then f is continuous. (5%)

(b) if f_n is integrable, then f is integrable. (5%)

I.2. Show that the following functions are uniformly continuous or not on their domain.

(a) $f(x) = \frac{1}{x^2}$, domain $(f) = \{x \in \mathbb{R} : x > 0\}$. (5%)

(b) $h(x) = \frac{1}{1+x^2}$, domain $(h) = \mathbb{R}$. (5%)

I.3. Let $f : D \to \mathbb{R}^m$ be an uniformly continuous function, where $D \subseteq \mathbb{R}^n$, $m, n \geq 1$. If $\{x_k\}$ is Cauchy sequence in D, show that $\{f(x_k)\}$ is a Cauchy sequence. (5%)

(b) Suppose $f : (0, 1) \to \mathbb{R}$ is uniformly continuous on $(0, 1)$. Show that f can be defined at $x = 0$, and $x = 1$ so that f is continuous on $[0, 1]$. (5%)

I.4.

(a) Show that $\tan^{-1} x = \sum\limits_{k=0}^{+\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \quad \forall x \in [-1, 1]$. (5%)

(b) Show that $\pi = 4 \sum\limits_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$. (5%)

I.5.

(a) Let $S = \{(x, t) : a \leq x \leq b, c \leq t \leq d\}$ and $f : S \to \mathbb{R}$ be a continuous function. Define $F : [c, d] \to \mathbb{R}$ by $F(t) = \int_a^b f(x, t)\, dx$. Show that F is continuous. (5%)

(b) In (a), if f and its partial derivative $\frac{\partial f}{\partial t}$ are continuous on S, then F is differentiable on $[c, d]$ and

$$F'(t) = \int_a^b \frac{\partial f(x, t)}{\partial t}\, dx. \qquad (5\%)$$

Part II.

II.1. Let $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ always solve the linear system $Ax = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, for any scalar c. Find A. (5%)

II.2. Without evaluating the following matrix A, find bases for the four fundamental subspaces, that is, the column space, the row space, the null space, and the left null space of the matrix A, respectively.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

(5%)

II.3. Let $T : \wp_2(\mathbb{R}) \longrightarrow \wp_2(\mathbb{R})$ be defined by $T(f) = f(0) + f(1)(x + x^2)$, where $\wp_2(bbr)$ is the set of real polynomials of degree ≤ 2.

(i) Show that T is a linear transformation from $\wp_2(\mathbb{R})$ to $\wp_2(\mathbb{R})$.

(ii) Show that T is diagonalizable. (7.5%)

II.4. Let $T : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$ with $T(M) = M^\top$ for $M \in \mathbb{R}^{n\times n}$.

(i) Show that T is a linear transformation from $\mathbb{R}^{n\times n}$ to $\mathbb{R}^{n\times n}$.

(ii) Show that there does not exist a matrix A such that $T(M) = AM$.

(iii) Does the result of 4. (ii) mean that the linear transformation T doesn't come from a matrix? Does this mean that the representation theorem fails in this example? (Note that the representation theorem concludes that each one linear transformation can be represented by a unique matrix up to the considering bases.) (10%)

II.5. Let $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ and $F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}$ be the 4th Fourier matrix, where $i = \sqrt{-1}$.

(i) Show that F is an eigenmatrix of P, i.e., there is a diagonal matrix $\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{bmatrix}$ such that $P = F\Lambda F^{-1}$.

(ii) Let $C = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{bmatrix}$ be a circulant matrix. Show that $C = c_0 I + c_1 P + c_2 P^2 + c_3 P^3$ and find the eigenvalues of C. (10%)

II.6. Find the maximum value of $R(x_1, x_2) = \frac{(x_1^2 - x_1 x_2 + x_2^2)}{(x_1^2 + x_2^2)}$, for $x_1, x_2 \in \mathbb{R}$ and $x_1^2 + x_2^2 \neq 0$. (5%)

II.7. Let T be the indefinite integral operator

$$(Tf)(x) = \int_0^x f(t)dt$$

on the space of continuous functions on the interval $[0, 1]$. Is the space of polynomial functions invariant under T? The space of differentiable functions? The space of functions which vanish at $x = \frac{1}{2}$? (7.5%)