91 academic year

Part I.

1.

(i) Let $a = 0.9999\ldots = 0.\bar{9}$ and $b = 1$. Is $a < b$? Or is $a = b$? Explain your answer.

(ii) Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = xy$. Show that f is a continuous function by using the ϵ-δ language. (10%)

2.

(i) Let $f: [0,\infty) \to \mathbb{R}$ be a bounded continuous function that is improper Riemann integrable on the interval $[0,\infty)$. Is $\lim_{x\to\infty} f(x) = 0$? Why or why not?

(ii) Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function. Is $f(K)$ closed when K is a closed subset of $\mathbb{R}$? Is $f(M)$ bounded and closed when M is a bounded and closed subset of $\mathbb{R}$? Why or why not? (10%)

3.

$$f(x) = \begin{cases} x + 2x^2 \sin(1/x), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Show that $f'(0) \neq 0$ but that f is not locally invertible near 0. Why does this not contradict the inverse function theorem? (10%)

4. Let a_k be a sequence of real numbers. Suppose that the series $\sum_{k=0}^{\infty} a_k$ converges.

(i) Does the power series $\sum_{k=0}^{\infty} a_k x^k$ converge uniformly on the interval $[0,1]$? Why or why not?

(ii) Is $\lim_{x\to 1^-} \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k$? Why or why not? (10%)

5. Define $\rho: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by $\rho((x_1,y_1),(x_2,y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$.

(i) Check that $(\mathbb{R}^2, \rho)$ is a metric space.

(ii) Let d be the usual metric of $\mathbb{R}^2$ i.e., $d((x_1,y_1),(x_2,y_2)) = ((x_1-x_2)^2 + (y_1 - y_2)^2)^{1/2}$. Is an open set in $(\mathbb{R}^2, d)$ also an open set in $(\mathbb{R}^2, \rho)$? Why or why not? (10%)

Part II.

6. Does there exist 3×3 matrices A and B satisfying $AB - BA = I$ (I is the identity matrix)? Why? (8%)

7. Give an example of two 3×3 matrices which are similar, but not unitarily equivalent, and explain your answer. (8%)

8. Let V be a vector space and $T: V \to V$ be linear. Show that if $T^2 = T$, then $V = \ker(T) \oplus \operatorname{ran}(T)$, the direct sum of kernel and range of T. (8%)

9. Suppose A, B and C are 3×3 matrices. Prove that

$$\det\left(\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}\right) = \det(A)\det(C).$$

(8%)

10. Let $(V, <,>)$ be a finite-dimensional inner product space over $\mathbb{C}$ and

$F : V \to \mathbb{C}$ be linear. Show that there exists an unique $y \in V$ such that $F(x) = \langle x, y \rangle$ for all $x \in V$. (8%)

11. Evaluate A^{100}, where $A = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}$. (10%)