89 academic year

Show all works

1. Let $U = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \,\big|\, x_1 + x_2 + x_3 + x_4 = 0,\ x_1 - x_2 - x_3 + 3x_4 = 0\}$ and V be the vector subspace generated by the vector $(0, 1, 0, 0)$ and U.
   (a) Find an orthonormal basis of U. (5%)
   (b) Find an orthonormal basis of V. (5%)

2. Let $x \in \mathbb{R}$, discuss the rank of the matrix $\begin{pmatrix} x & 0 & 0 & 1 \\ 1 & x & 1 & 0 \\ 0 & 1 & x & 1 \\ 1 & 0 & 1 & x \end{pmatrix}$. (10%)

3. Let $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$.
   (a) Find the characteristic polynomial of A. (5%)
   (b) Find the minimal polynomial of A. (5%)
   (c) If $f(X) = X^5 - 7X^4 + 9X^3 + 9X^2 - 7X + 8$, find $f(A)$. (5%)
   (d) Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. (5%)

4. Define $f(x) = \left(\int_0^x e^{-t^2}\, dt\right)^2$ and $g(x) = \int_0^1 \frac{e^{-x^2(t^2+1)}}{t^2+1}\, dt$.
   (a) Show that $f'(x) + g'(x) = 0$, for all x and deduce that $f(x) + g(x) = \frac{\pi}{4}$. (5%)
   (b) Use (a) to prove that $\int_{-\infty}^{\infty} e^{-t^2}\, dt = \sqrt{\pi}$. (5%)

5. Let f be a positive continuous function in $[a, b]$. Let M be the maximal value of f on $[a, b]$. Show that $\lim_{n\to\infty} \left(\int_a^b f(x)^n\, dx\right)^{1/n} = M$. (10%)

6. Suppose that $a_n > 0$, $s_n = a_1 + a_2 + \cdots + a_n$, and $\sum a_n$ diverges.
   (a) Prove that $\sum \frac{a_n}{1 + a_n}$ diverges. (10%)
   (b) What can we say about $\sum \frac{a_n}{1 + na_n}$? (10%)

7. Determine all real values of x for which the following series converges:

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \frac{\sin nx}{n}.$$

(10%)

8. Let (R^2, ρ) be a metric space where $R^2 = \{x = (x_1, x_2) \mid x_1, x_2 \in R\}$ and $\rho(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.
   Show that the set $S = \{x \in R^2 \mid \sqrt{x_1^2 + x_2^2} < 1\}$ is an open and connected set. (10%)