88 academic year

Show all works

1.
   (a) [10%] Show that the integral $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$.
   (b) [5%] Evaluate the integral $\int_0^\infty \frac{\sin x \cos xy}{x} dx$.

2. [10%] Let (x, y, z) and (ρ, θ, ϕ) be the rectangular coordinates and the spherical coordinates, respectively, for $\mathbf{R}^3$. Compute $\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)}$.

3. [15%] Explain the identity

$$\frac{1}{1+x^2} = \Sigma_{n=0}^{\infty}(-1)^n x^{2n}$$

   why the left side is defined on $\mathbf{R}^1$ while the right side is only defined on the interval $-1 < x < 1$?

4. [15%] Suppose that the series $\Sigma_{n=1}^{\infty} a_n$ converges and for each $a_n \geq 0$. Discuss the convergence of the series

$$\Sigma_{n=1}^{\infty} \sqrt{a_n} n^{-p}, \quad p \in \mathbf{R},$$

   on which interval the series converges and on which interval the series may or may not diverge. If it is in the latter case, please give examples.

5. Let $A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ a & 3 & 0 & 0 \\ b & c & -2 & 0 \\ d & e & f & 5 \end{pmatrix}$, where $a, b, c, d, e, f \in \mathbf{C}$.
   (a) [4%] Find all possible characteristic and minimal polynomials for A.
   (b) [8%] Find all possible Jordan forms of A.
   (c) [3%] Find all possible diagonal matrix that are similar to A.

6. Let $A \in M(n, \mathbf{C})$, set of all $n \times n$ matrices with complex entries, such that $A^* = -A$, and let $B = e^A$. (Recall that the joint matrix, A^*, of the matrix A is given by $(Ax, y) = (x, A^* y)$; B is unitary if $BB^* = I$.) Show that
   (a) [5%] $\det B = e^{trA}$;
   (b) [5%] $B^* = e^{-A}$;
   (c) [5%] B is unitary.

7. [15%] Evaluate the area enclosed by the curve $13x^2 + 10xy + 13y^2 - 72 = 0$. (Hint: use Green's Theorem.)