八十五學年度碩士班推薦入學考試 甄試 共3頁之1

Part 1

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that if f is an open mapping (i.e. $f(V)$ is open whenever V is open), then f is monotonic. (10%)

2. Let f be defined on $[0,1]$ as follows: $f(0) = 0$; if $2^{-n-1} < x \leq 2^{-n}$, then $f(x) = 2^{-n}$ for $n = 0, 1, 2, \ldots$.

   (a) Give a reason why $\int_0^1 f(x)dx$ exists. (10%)

   Let $F(x) = \int_0^x f(t)dt$ and $A(x) = 2^{-[-\ln x/\ln 2]}$, where $[y]$ is the greatest integer in y.

   (b) Show that $F(x) = xA(x) - \frac{1}{3}A(x)^2$ for $x = 2^{-n}$, $n = 0, 1, 2, \ldots$, (3%)

   (c) Show that the equality in (b) holds for $0 < x \leq 1$. (5%)

3. Prove that the series $\sum (-1)^n \frac{x^2 + n}{n^2}$ converges uniformly in every bounded interval, but does not converge absolutely for any value of x. (10%)

4. Find the maximum of $(x_1 \cdots x_n)^2$ under the restriction $x_1^2 + \cdots + x_n^2 = 1$. Use the result to derive the inequality, $(a_1 \ldots a_n)^{\frac{1}{n}} \leq \frac{a_1 + \cdots + a_n}{n}$ for positive real numbers $a_1, \ldots, a_n$. (10%)

5. (a) Show that the area of a region D to which Green's theorem applies may be given by $A(D) = \frac{1}{2}\int_{\partial D} xdy - ydx$. (6%)

   (b) Allpy this to find the area bounded by the ellipse $x = a\cos\theta$, $y = b\sin\theta$, $0 \leq \theta \leq 2\pi$. (4%)

Part II

§1. Instruction

Read carefully the definitions and terminology given in the following section before you work on any of the problems.

§2. Definitions and Terminology

In what follows, we fix F for a field and V for a vector space over F. The field of real numbers is denoted by $\mathbb{R}$. The dimension of the vector space V will be denoted by $\dim V$. Let $L(V,V)$ be the set of all linear transformations from V to V and let $M(n,F)$ be the set of all $n \times n$ matrices, where n a positive integer.

For any linear transformation $f \in L(V,V)$, the *kernel* of f (denoted by $\ker f$) is the set of all $v \in V$ such that $f(v) = 0$; the *nullity* of f is then defined to be the dimension of $\ker f$.

We call a linear transformation $f \in L(V,V)$ *nilpotent* if there exists some positive integer $n \in \mathbb{N}$ such that $f^n(v) = 0$ for all $v \in V$. A subspace W of V is called *cyclic with respect to* f if it is spanned by w_0, $f(w_0)$, $f^2(w_0)$, ..., $f^{n-1}(w_0)$ for some $w_0 \in W$.

§3. Problems

(1) (i) Let $C^\infty(\mathbb{R})$ denote the vector space of all real valued functions on $\mathbb{R}$ which has derivatives of every order. Consider the differential operator (a linear transformation) $D : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ given by 5%

$$D(y) = y'' + py' + qy,$$

where p and q are real constants. Show that $e^{\lambda x}$ lies in the kernel of D if and only if λ is a root of the polynomial

$$p(t) = t^2 + pt + q.$$

(ii) Find two linearly independent solutions to the homogeneous differential equation 3%

$$y'' - 5y' + 4y = 0.$$

(2) (i) Let $v_1, v_2, \ldots, v_n$ be a basis of V and let $T \in L(V,V)$. Prove that T is nilpotent if and only if there exist positive integers $r_1, r_2, \ldots, r_n$ such that $T^{r_j}(v_j) = 0$ for $j = 1, 2, \ldots, n$. 6%

(ii) Let $T \in L(V,V)$ be nilpotent, and W a one-dimensional subspace of V. Show that W is cyclic with respect to T if and only if W lies in the kernel of T. 6%

(3) Two matrices A and B are said to commute if $AB = BA$. Find the set of all $A \in M(n,F)$ which commute with all the diagonal matrices in $M(n,F)$. 8%

(4) Let $f, g \in L(V, V)$. Suppose that $f \circ g = id_V$, the identity mapping of V. Is f a bijective mapping? If yes, prove it; otherwise, give a counter example. (Hint. Consider the dimension of V.) 10%

(5) Suppose that $\dim V = n$ and let $f : V \to F$ be a nonzero linear transformation from V to F. Prove that the nullity of f is $n - 1$. 6%

(6) Let $T \in L(V, V)$. Prove that there exists a nonzero linear transformation $S \in L(V, V)$ such that $TS = 0$ if and only if there exists a nonzero vector $v \in V$ such that $T(v) = 0$. 6%