应數所

國立成功大學八十四學年度碩士班甄試入學考試(基礎数學試題) 共 1 頁 第 1 頁

Work each of your problems (Parts I ,II).

Part I. Advanced Calculus

1. Suppose $f : (a, b) \to \mathbb{R}$ is uniformly continuous. Show that there is l in $\mathbb{R}$ such that $\lim_{x \to b^-} f(x) = l$. (10%)

2. Does the series $\sum_{k=0}^{\infty} \frac{x^2}{(1+x^2)^k}$ converge pointwise on $\mathbb{R}$? (5%)
   Does it converge uniformly on $\mathbb{R}$? (5%)

3. Let $f : [a, b] \to \mathbb{R}$ be continuously differentiable with $f(a) = f(b) = 0$ and $\int_a^b (f(x))^2 dx = 1$. Evaluate $\int_a^b t f(t) f'(t) dt$. (10%)

4. Suppose $f : [0, \infty) \to [0, \infty)$ is Riemann integrable. Show that if f is uniformly continuous on $[0, \infty)$, then $\lim_{x \to \infty} f(x) = 0$. (10%)
   Can "uniform continuity" be replaced by "continuity"? (10%)

Part II. Linear Algebra

1. Let V be an inner product space over a field F and $T, S : V \to V$ be linear.
   (a) Show that if $< Tx, y > = < Sx, y >$ for all $x, y \in V$, then $T = S$. (4%)
   (b) Show that if $F = \mathbb{C}$ and $< Tx, x > = < Sx, x >$ for all $x \in V$, then $T = S$. (10%)
   (c) Does the conclusion in (b) hold when $F = \mathbb{R}$? Justify your answer. (6%)

2. Give an example of two square matrices A and B such that they have the same characteristic polynomial and minimal polynomial, but A is not similar to B. (10%)

3. Let V be a finite-dimensional vector space, $T : V \to V$ be linear, $N(T)$ be the null space of T and $R(T)$ be the range of T. Show that
   (a) if $rank(T) = rank(T^2)$,then $V = R(T) \oplus N(T)$ (the direct sum of $R(T)$ and $N(T)$) ; (10%)
   (b) there exists a positive integer k such that $V = R(T^k) \oplus N(T^k)$. (10%)