國立成功大學 113 學年度「碩士班」甄試入學考試

高等微積分

Advanced Calculus

October 19, 2023

1. *(15 points)* Let $f : [0,1] \to \mathbb{R}$ be a function defined by

$$f(x) := \begin{cases} \frac{1}{k} & \text{when } x \in \mathbb{Q} \text{ and } x = \frac{q}{2^k} \text{ for some } q, k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

   Prove that f is continuous at every irrational point.

2. **(a)**. *(10 points)* Determined whether the following series is convergent or not:

$$\sum_{n=1}^{\infty} 5^{-n+(-1)^n}.$$

   (b). *(10 points)* Prove that the power series $\sum_{n=0}^{\infty} a_n x^n$ converges on $(-R, R)$ when $\sum_{n=0}^{\infty} R^{2n} a_n^2$ converges.

3. *(15 points)* Let $X \subset \mathbb{R}^2$ be an open set and $f : X \to \mathbb{R}$ be a continuous function with $f(x) \neq 0$ for all $x \in X$. Prove that either $f(x) > 0$ for all $x \in X$, $f(x) < 0$ for all $x \in X$ or X is disconnected[1].

4. *(20 points)* Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of positive, Riemann integrable functions defined on $[0,1]$ and $g_n(x) \leq 1$ for all $x \in [0,1]$, $n \in \mathbb{N}$. Prove that the sequence

$$\left\{ f_n(x) := \int_0^x g_n(s)ds \right\}_{n\in\mathbb{N}}$$

   has a subsequence which is uniformly convergent on $[0,1]$.

5. *(15 points)* Let $f : [0,1] \to \mathbb{R}$ be a Riemann integrable function. Prove that f^3 is also Riemann integrable.

6. *(15 points)* Let $S := \{(x,y,z) \in \mathbb{R}^3 | z = 7 - x^2 - y^2, z \geq 3\}$, $\vec{n}$ be the outer normal of S and

$$V := (z^2, -3xy, x^3y^3)$$

   be a vector field defined on $\mathbb{R}^3$. Find

$$\int_S curl(V) \cdot \vec{n} ds.$$

[1] A set X is disconnected iff there exist two disjoint nonempty open sets A, B cover X and $A \cap X \neq \emptyset$, $B \cap X \neq \emptyset$.

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線性代數

- Show all your work and justify all your answers.
- $\mathbb{R}$ denotes the field of real numbers, and n denotes a positive integer.

1. (12 points) Let A be an $n \times n$ real matrix whose (i, j) entry is

$$A_{ij} = \begin{cases} j, & \text{if } i \leq j, \\ 0, & \text{otherwise,} \end{cases}$$

where $i, j = 1, \ldots, n$. Find the inverse of A.

2. (12 points) Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$. For $(x_1, y_1), (x_2, y_2) \in V$ and $a \in \mathbb{R}$, define

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 y_2) \quad \text{and} \quad a(x_1, y_1) = (ax_1, y_1).$$

Is V a vector space over $\mathbb{R}$ with these operations?

3. (15 points) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the function defined by

$$T(x, y, z) = (2x - y, 3y - 2z, x + y - z)$$

for $(x, y, z) \in \mathbb{R}^3$. Prove that T is a linear transformation. Is T one-to-one?

4. (15 points) Let $V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid 2x_1 - x_2 + 3x_3 - x_4 = 0\}$. Find a basis β for V such that $(2, 1, -1, 0) \in \beta$.

5. (15 points) Let V be the real vector space of all $n \times n$ real matrices, and let $A \in V$. Suppose that W is the subspace of V spanned by the set $\{A^i \mid i \text{ is a non-negative integer}\}$, where A^0 is defined to be the $n \times n$ identity matrix. Prove that $\dim(W) \leq n$.

6. (15 points) Let V be a finite-dimensional complex inner product space, and let T be a positive definite linear operator on V. Prove that $T = S^*S$ for some invertible linear operator S on V. Here S^* denotes the adjoint of S.

7. (16 points) Find the Jordan canonical form of the real matrix $\begin{bmatrix} 4 & -3 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.