國立成功大學 112 學年度「碩士班」甄試入學考試

高等微積分

1. (a) Determine the limit of $f(x) = \dfrac{3x^2 - 5x + 4}{x+1}$ at $x = 2$ by δ-ϵ definition. (10%)

   (b) Determine whether $f(x)$ is uniformly continuous on $[0, \infty)$. (10%)

2. Show by definition that the interior of triangular region with vertices $(1,1)$, $(5,2)$, $(3,4)$ is an open set. (10%)

3. Denote x_n the positive root of the polynomial $f_n(x) = x^n + \cdots + x^2 + x - 1$. Show that the sequence $\{x_n\}$ is convergent and find the limit. (15%)

4. Given the fact that the following integral is convergent for all $p > 0$.

$$\int_1^{\infty} \frac{\sin x}{x^p}\, dx$$

   For what values of $p > 0$ is the integral convergent absolutely/conditionally? (15%)

5. (a) By observing the graph of $y = \frac{n}{1+n^2x^2}$ as n increases and evaluating its integral on $(-\infty, \infty)$, find the value the following limit. (10%)

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{n\, e^{\cos x}}{1+n^2x^2}\, dx$$

   (b) Justify the convergence by N-ϵ definition. (10%)

6. (a) Evaluate the limit. (10%)

$$\lim_{n\to\infty} \left(\frac{3^n + 4^n + 5^n}{3}\right)^{\frac{1}{n}} + \left(\frac{3^{\frac{1}{n}} + 4^{\frac{1}{n}} + 5^{\frac{1}{n}}}{3}\right)^n$$

   (b) Evaluate the limit. (10%)

$$\lim_{n\to\infty} \left(\int_0^1 e^{nx(1-x)}\, dx\right)^{\frac{1}{n}} + \left(\int_0^1 e^{\frac{x(1-x)}{n}}\, dx\right)^n$$

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線性代數

1. (10 points) Let A, B be two $m \times n$ matrix. Show that $|\operatorname{rank}(A) - \operatorname{rank}(B)| \leq \operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$.

2. (16 points) Let A be an $n \times n$ matrix and $r_k = \operatorname{rank}(A^k)$.

   (a). Show that $\lim_{k\to\infty} r_k$ exist.

   (b). If $r_3 \neq r_4$, Is A diagonalizable? Show your answer.

3. (8 points) Let $A = [a_{ij}]$ be an $n \times n$ matrix satisfying the condition that each a_{ij} is either equal to 1 or to -1. Show that $\det(A)$ is an integer multiple of 2^{n-1}.

4. (16 points) Let S, T be linear operator on V such that $S^2 = S$. Show that the range of S is invariant under T if and only if $STS = TS$. Show that both the range and null space of S are invariant under T if and only if $ST = TS$.

5. (20 points) Define a real vector space $V = \{f(x) \mid f(x) = ax^2 + bx + c, a, b, c \in \mathbb{R}\}$, with inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$.

   (a). Find an orthonormal basis for V.

   (b). Using (a), find $f \in V$ to maximize $f(\frac{1}{2})$ subject to the constraint $\langle f, f\rangle = 1$.

6. (16 points) Let $A = \begin{bmatrix} -2 & 3 & 1 \\ 0 & a & 3 \\ 0 & -3 & 4-a \end{bmatrix}$. Find the condition of a such that A is diagonalizable over real number.

7. (14 points) Let A be an $n \times n$ real symmetric matrix. Show that the matrix $A^2 + A + I$ is positive-definite.