國立成功大學111學年度「碩士班」研究生甄試入學考試

高等微積分

Throughout the exam, the Euclidean spaces $\mathbb{R}^n$ are all equipped with usual Euclidean metric $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$.

1. (10 points) Evaluate the line integral

$$\int_C \mathbf{V} \cdot d\mathbf{r},$$

where the vector field is given by

$$\mathbf{V} = \langle 2y + xe^{x^2}, 4x + e^{y^2} \cos y \rangle,$$

and C is the path staring from $(1, 0)$ to $(0, -1)$ by a *counterclockwise* circular path, and then from $(0, -1)$ to $(0, 0)$ by a straight line.

2. For a continuous function $f : [a, b] \to \mathbb{R}$ that is differentiable on (a, b),
   (a) (8 points) State and prove the *Rolle's Theorem* for f.
   (b) (7 points) State and prove the *Mean Value Theorem* for f.

3. (10 points) Given the fact that $\mathbb{R}$ is complete, prove the *Monotone Convergence Theorem*:

Any bounded monotonic sequence in $\mathbb{R}$ is convergent.

4. Consider the usual Euclidean space $\mathbb{R}^n$,
   (a) (8 points) If $E \subset K \subset \mathbb{R}^n$, where E is closed and K is compact, then E is compact.
   (b) (8 points) Prove that if $\{K_\alpha\}_{\alpha \in A}$ is any family of compact subsets of $\mathbb{R}^n$, then

$$\bigcap_{\alpha \in A} K_\alpha$$

is also compact.

5. (15 points) Prove that the sequence of functions $\{f_n\}$ on $[0, 1]$ given by

$$f_n(x) = nx(1 - x^2)^n$$

pointwise converges to a function $f(x)$ but does not converge uniformly.

6. Given two metric spaces (E, d_E) and (F, d_F),

   (a) (8 points) If $f : E \to F$ is continuous and E is compact, then f is uniformly continuous.

   (b) (8 points) Use part (a) to prove that

   $$f(x) = e^{\cos^2 x \sin x}$$

   is uniformly continuous on $\mathbb{R}$.

7. (a) (4 points) State the *Inverse Function Theorem* on $\mathbb{R}^n$.

   (b) (4 points) State the *Implicit Function Theorem* on $\mathbb{R}^{n+m}$.

   (c) (10 points) These two theorems are in fact equivalent. Prove *either* one of the implications. (i.e. prove either (a) $\Rightarrow$ (b) or (b) $\Rightarrow$ (a)).

國立成功大學 111 學年度「碩士班」研究生甄試入學考試

線性代數

In this test, all vector spaces are finite dimensional over $\mathbb{C}$.

1. (15 points) Let T be a linear operator on a vector space V. Prove that T is diagonalizable if and only if its minimal polynomial is square-free.

2. (15 points) Let V be a vector space. A linear operator S on V is semisimple if for every S-invariant subspace W of V there exists an S-invariant subspace W' of V such that $V = W \oplus W'$. Prove that every diagonalizable operator on V is semisimple, and deduce that every linear operator T on V can be decomposed uniquely as $T = S + N$, where S is semisimple, N is nilpotent, and $SN = NS$.

3. (15 points) Let T and U be normal operators on an inner product space V such that $TU = UT$. Prove that $UT^* = T^*U$, where T^* is the adjoint of T.

4. (15 points) Let T and U be Hermitian operators on an inner product space $(V, \langle \cdot, \cdot \rangle)$ such that $\langle T(x), x \rangle > 0$ for all nonzero $x \in V$. Prove that UT is diagonalizable and has only real eigenvalues.

5. (15 points) Find the total number of distinct equivalence classes of congruent $n \times n$ real symmetric matrices and justify your answer.

6. (15 points) Let A be an $n \times n$ complex matrix, t be a variable, and I be the identity matrix. Prove that
$$\det(I - tA) = \exp\left(-\sum_{i \geq 1} \frac{\operatorname{tr}(A^i) t^i}{i} \right).$$

7. (10 points) Let $A = (a_{i,j})$ be a $2n \times 2n$ matrix such that $A^T = -A$. The Phaffian of A is defined as
$$\operatorname{pf}(A) := \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)},$$
where S_{2n} is the symmetric group of order $2n$ and $\operatorname{sgn}(\sigma)$ is the signature of σ. Prove that for any $2n \times 2n$ matrix B,
$$\operatorname{pf}(BAB^T) = \det(B) \operatorname{pf}(A).$$