國立成功大學 109 學年度「碩士班」研究生甄試入學考試

高等微積分

1. (15%) Suppose that E is a compact subset of $\mathbb{R}$ and that $f : \mathbb{R} \to \mathbb{R}$. Prove that if f is continuous on E, then f is uniformly continuous on E.

2. (a) (5%) State the Implicit Function Theorem.

   (b) (10%) Prove that there exist functions $u(x,y), v(x,y)$, and $w(x,y)$, and an $r > 0$ such that u, v, w are continuously differentiable and satisfy the equations

$$\begin{aligned} u^5 + xv^2 - y + w &= 0 \\ v^5 + yu^2 - x + w &= 0 \\ w^4 + y^5 - x^4 &= 1 \end{aligned}$$

   on $B_r(1,1)$, and $u(1,1) = 1, v(1,1) = 1, w(1,1) = -1$.

3. (a) (10%) Prove that if $f : [0,1] \to \mathbb{R}$ is continuous and $\int_0^1 |f(x)|dx = 0$, then $f(x) = 0$ for all $x \in [0,1]$.

   (b) (5%) State the Weierstrass Approximation Theorem.

   (c) (10%) Prove that if $f : [0,1] \to \mathbb{R}$ is continuous and

$$\int_0^1 f(x)x^k dx = 0 \quad \text{for } k = 0, 1, \cdots,$$

   then $f(x) = 0$ for all $x \in [0,1]$.

4. (a) (5%) State the Arzela-Ascoli Theorem.

   (b) (10%) Let $f_n : [0,1] \to \mathbb{R}$ be continuous and be such that $f_n(0) = 0$ for every $n \in \mathbb{N}$. Suppose that the derivatives f_n' exist and are uniformly bounded on $(0,1)$. Prove or disprove that f_n has a uniformly convergent subsequence.

5. (15%) Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \le \frac{1}{2}|x - y|$ for all $x, y \in \mathbb{R}$. Prove that f has a unique fixed point.

6. (15%) Let $f_n : [0,1] \to \mathbb{R}$ be continuous for every $n \in \mathbb{N}$. Suppose that $\{f_n\}$ converges uniformly to f on $[0,1]$ and that $\{x_n\}$ is a sequence in $[0,1]$ converging to a point $x \in [0,1]$. Prove or disprove that $\lim_{n\to\infty} f_n(x_n) = f(x)$.

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線性代數

1. Find e^A, where $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$. (15 points)

2. Let $T_j : \mathbb{R}^2 \to \mathbb{R}^2$, $j = 1, 2$, be a rotation by some angle θ_j about some point $x_j \in \mathbb{R}^2$. Show that if $\theta_1 + \theta_2 \notin \{2k\pi \,|\, k \in \mathbb{Z}\}$, then the composition T_2T_1, is also a rotation about some point. (15 points)

3. Let A be a real skew-symmetric matrix, that is, $A^t = -A$. Prove the following statements.

   (a) Each eigenvalue of A is either 0 or a purely imaginary number. (10 points)

   (b) The rank of A is even. (10 points)

4. Let $C([-\pi, \pi])$ be the space of real continuous functions on $[-\pi, \pi]$ with inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$. Find an orthonormal basis for the subspace $W = \operatorname{span}(1, x, \sin x)$. (15 points)

5. Let $M_{2\times 2}$ be the space of 2×2 real matrices. Consider the linear operator S on $M_{2\times 2}$ defined by

$$S(X) = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} X + X \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix},$$

   where $a, b, c, d \in \mathbb{R}$.

   (a) Write down the representative matrix of S with respect to the basis

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}. \text{ (10 points)}$$

   (b) Give the necessary and sufficient condition on a, b, c, d so that S is invertible. (10 points)

6. Let A be an $n \times n$ (real or complex) matrix. Show that if A is nilpotent (i.e. $A^k = 0$ for some $k \in \mathbb{N}$), then $I - A$ is invertible, where I is the $n \times n$ identity matrix. (15 points)