國立成功大學 105 學年度「碩士班」研究生甄試入學考試

【基礎數學】: Part I. 高等微積分

1. Entrance Exam

(1) (15 Points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$f(x,y) = \begin{cases} \frac{x}{y} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}.$$

(a) (5 Points) Use definition to show that $f_x(0,0)$ and $f_y(0,0)$ exist.
(b) (5 Points) Prove that f is not continuous at $(0,0)$.
(c) (5 Points) Is f differentiable at $(0,0)$? Explain.

(2) (15 Points) Let $a > 0$. The following iterated integral can be rewritten as a double integral:

$$\int_{-a}^{a} \left(\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)} dy \right) dx = \iint_D e^{-(x^2+y^2)} dA.$$

Sketch the region D and evaluate the iterated integral by computing the double integral.

(3) (15 Points)
(a) (7 Points) State Rolle's Theorem.
(b) (8 Points) Let $f : (0,\infty) \to \mathbb{R}$ be a continuously differentiable function. Suppose that the equation

$$\frac{x^2}{2} + \ln x = f(x)$$

has at least two distinct solutions (on $(0,\infty)$). Show that there exists a positive real number t such that $f'(t) \geq 2$.

(4) (15 Points)
(a) (7 Points) State the Bolzano-Weierstrass Theorem (in $\mathbb{R}^n$.)
(b) (8 Points) Prove or disprove that the sequence of real numbers $\{a_n\}$ defined by

$$a_n = e^{\sin n}, \quad n \in \mathbb{N}$$

has a convergent subsequence.

(5) (15 Points) For each vector $x \in \mathbb{R}^n$, we define its Euclidean norm to be

$$\|x\|_{\mathbb{R}^n} = \sqrt{x_1^2 + \cdots + x_n^2},$$

where $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. For any two vectors $x, y \in \mathbb{R}^n$, we define their inner product to be

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i.$$

Let $a_1, \cdots, a_m$ be unit vectors in $\mathbb{R}^n$, i.e. $\|a_i\|_{\mathbb{R}^n} = 1$ for all $1 \leq i \leq n$. Define a function $T : \mathbb{R}^n \to \mathbb{R}^m$ by

$$T(x) = (a_1 \cdot x, \cdots, a_m \cdot x), \quad x \in \mathbb{R}^n.$$

(a) (7 Points) Prove that

$$\|T(x)\|_{\mathbb{R}^m} \leq \sqrt{m}\|x\|_{\mathbb{R}^n} \text{ for any } x \in \mathbb{R}^n$$

(b) (8 Points) Prove that T is uniformly continuous on $\mathbb{R}^n$ via $\epsilon - \delta$ language.

(6) (25 Points) Let $X = C([0,1])$ be the space of real valued continuous functions on $[0,1]$ equipped with the norm

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|.$$

For each $f, g \in X$, we set $d(f,g) = \|f-g\|_\infty$. It is well known that (X,d) is a complete metric space, i.e every Cauchy sequence in X is convergent.

For each $n \in \mathbb{N} \cup \{0\}$, define $f_n : [0,1] \to \mathbb{R}$ inductively by $f_0 = 0$ and

$$f_{n+1}(x) = 1 + \int_0^x t f_n(t)dt, \quad n \geq 0.$$

Then $f_n \in X$ for all $n \in \mathbb{N}$.

(a) (7 Points) Prove that

$$\|f_{n+1} - f_n\|_\infty \leq \frac{1}{2}\|f_n - f_{n-1}\|_\infty, \quad \text{for all } n \in \mathbb{N}.$$

(b) (9 Points) Use completeness of X to show that $\{f_n : n \in \mathbb{N}\}$ is uniformly convergent. (Hint: use (a) to show that $\{f_n\}$ is a uniform Cauchy sequence.)

(c) (9 Points) Let f be the uniform limit of $\{f_n : n \in \mathbb{N}\}$ in X i.e.

$$f = \lim_{n\to\infty} f_n \text{ in } X.$$

Show that f is continuously differentiable on $[0,1]$ and solve for f.

【基礎數學】：Part II. 線性代數

Entrance exam for master degree program: Linear Algebra

1. (20 points) Consider the 5×5 real matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

   and the following problems concerning A.

   (a) Find an invertible matrix P such that PA is a row-reduced echelon matrix.

   (b) Find a basis for the row space W of A.

   (c) Find a basis for the vector space V of all 5×1 column matrices X such that $AX = \mathbf{0}$.

   (d) For what 5×1 column matrices Y does the equation $AX = Y$ has solutions?

2. (10 points) Let A be an $n \times n$ real matrix with transpose A^T. Prove that $\operatorname{rank}(A^T A) = \operatorname{rank} A$.

3. (10 points) Let V be the vector space of all real 3×3 matrices and let A be the diagonal matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

   Calculate the determinant of the linear transformation T on V defined by $T(X) = AX + XA$.

4. (20 points) Let A be an $n \times n$ orthogonal matrix, that is, A is a real $n \times n$ matrix with $A^T A = I$ where I is the $n \times n$ identity matrix.

   (a) Show that $\det A = \pm 1$.

   (b) Show that x and Ax have the same length for all $x \in \mathbb{R}^n$.

   (c) If λ is an eigenvalue of A, Prove that $|\lambda| = 1$.

   (d) If $n = 3$ and $\det A = 1$, prove that 1 is an eigenvalue of A.

5. (15 points) Show that all the eigenvalues of a real symmetric matrix are real, and that the eigenvectors are perpendicular to each other when they correspond to different eigenvalues.

6. (15 points) Consider the matrix

$$A = \begin{pmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{pmatrix}.$$

   Find a Jordan form J of A and an invertible matrix Q such that $A = QJQ^{-1}$.

7. (10 points) Show that every matrix is similar to its transpose.

This exam has 7 questions, for a total of 100 points.