【基礎數學】：Part I. 高等微積分

1. (10 %) Suppose f is a real, three times differentiable function on $[-1,1]$, such that

$$f(-1)=0, \quad f(0)=0, \quad f(1)=1, \quad f'(0)=0.$$

Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1,1)$.

2. (12 %) Let

$$f_m(x)=\lim_{n\to\infty}(\cos m!\pi x)^{2n} \quad (x\in\mathbb{R},\ m\in\mathbb{N}).$$

Show that $\{f_m\}$ converges to a function f in $\mathbb{R}$, but not uniformly.

3. (a) (10 %) Show that $\int_{-\infty}^{\infty} e^{-x^2}dx=\sqrt{\pi}$.

   (b) (8 %) Show that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, where $\Gamma(\alpha)=\int_0^{\infty} e^{-x}x^{\alpha-1}dx$ is the gamma function, $0<\alpha<\infty$.

4. (15 %) Let $f:[0,1]\to[0,1]$ be continuous on $[0,1]$ and differentiable on $(0,1)$. Assume that $f(0)=0$ and $f(1)=1$. Show that for each $n\in\mathbb{N}$, there are n distinct points $a_1, a_2, \cdots, a_n \in [0,1]$ such that

$$\frac{1}{f'(a_1)}+\frac{1}{f'(a_2)}+\cdots+\frac{1}{f'(a_n)}=n.$$

5. (a) (5 %) State the Weierstrass approximation theorem.

   (b) (10 %) Let $f:[0,1]\to\mathbb{R}$ be a continuous function such that $\int_0^1 f(x)x^n dx=0$ for all nonnegative integers. Prove or disprove that $f(x)=0$ for all $x\in[0,1]$.

6. (a) (7 %) State the Arzela-Ascoli theorem.

   (b) (8 %) Let

$$f_n(x)=\frac{x^2}{x^2+(1-nx)^2} \quad (0\leq x\leq 1,\ n=1,\ 2,\ 3,\ \ldots).$$

   Show that $\{f_n\}$ is not equicontinuous on $[0,1]$.

7. Define

$$f(x,y)=\begin{cases} \sin\left(\frac{y^2}{x}\right)\cdot\sqrt{x^2+y^2} & \text{if } x\neq 0 \\ 0 & \text{if } x=0. \end{cases}$$

   (a) (7 %) Show that the function $f:\mathbb{R}^2\to\mathbb{R}$ is continuous at $(0,0)$ and has directional derivatives in every direction at $(0,0)$.

   (b) (8 %) Show that there is no plane that is tangent to the graph of $f:\mathbb{R}^2\to\mathbb{R}$ at the point $(0,0,f(0,0))$.

國立成功大學104學年度「碩士班」研究生甄試入學考試

【基礎數學】：Part II. 線性代數

In what follows, $\mathbb{R}$ denotes the field of all real numbers, and $M_{n\times n}(\mathbb{R})$ denotes the vector space of all $n \times n$ real matrices.

1. Let $P_2(\mathbb{R})$ be the vector space of all polynomials of degree at most 2 with real coefficients. Suppose $T\colon M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ is the linear transformation defined by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 3c + (a-2b)x + dx^2.$$

   (a) (10%) Find a basis for the null space of T and determine the dimension of the range of T.

   (b) (10%) Let $\gamma = \{1, x, x^2\}$, which is the standard ordered basis for $P_2(\mathbb{R})$. Find an ordered basis β for $M_{2\times 2}(\mathbb{R})$ such that the matrix representation $[T]_\beta^\gamma$ of T in β and γ is

$$\begin{pmatrix} 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

2. Let n be a positive integer, and let $S_i = \{A \in M_{n\times n}(\mathbb{R}) \mid A^t = (-1)^i A\}$ for $i = 1, 2$. Here A^t denotes the transpose of A.

   (a) (6%) Prove that S_i is a subspace of $M_{n\times n}(\mathbb{R})$ for $i = 1, 2$.

   (b) (12%) Prove that $M_{n\times n}(\mathbb{R})$ is the direct sum of S_1 and S_2.

3. Consider the real matrix $A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & -1 & -1 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$.

   (a) (8%) Find the characteristic polynomial for A.

   (b) (12%) Find the minimal polynomial for A. Is A similar to a diagonal matrix? *Justify your answer.*

4. (12%) Let V be a finite-dimensional inner product space whose inner product is denoted by $\langle \cdot\,, \cdot \rangle$, and let T be a self-adjoint operator on V (that is, T is equal to its adjoint T^*). Prove that if $\langle v, T(v) \rangle = 0$ for all $v \in V$, then T is the zero linear operator.

5. (18%) Let T be a linear operator on a finite-dimensional complex vector space V. Suppose W is a T-invariant subspace of V and $W \neq V$. Prove that there exists a vector $v \in V \setminus W$ such that $T(v) - \lambda v \in W$ for some eigenvalue λ of T.

6. (12%) Let A be a 9×9 real matrix such that $A^6 + A^3 = A^5 + A^4$. Is A similar over $\mathbb{R}$ to a upper triangular matrix? *Justify your answer.*