國立成功大學 103 學年度「碩士班」研究生甄試入學考試

【基礎數學】：Part I. 高等微積分

advanced calculus

1. Prove that $\lim_{x\to\infty}(x^5 - 3x^4 - x^2 + 1)^{-1} = 0$. (20 points)

2. Define a function f by

$$f(x) = \begin{cases} x \cdot \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Prove or disprove that f is uniformly continuous on $\mathbb{R}$. (20 points)

3. Let f be a smooth function on $(-1, 1)$ and $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $x \in (-1, 1)$. Let $\{x_m\}$ be a sequence with $x_m \neq 0$ for all $m \in \mathbb{N}$. Assume that $\{x_m\}$ converges to zero with $f(x_m) = 0$. Show that $f = 0$ on $(-1, 1)$. (20 points)

4. Let f be continuous on $[0, 1]$ and suppose that $f(0) = 0$. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x^n) = 0.$$

(20 points)

5. Evaluate the integral $\int_0^\infty e^{-x^2} dx$. (20 points)

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【基礎數學】：Part II. 線性代數

Notation

- n: a positive integer
- $M_{n\times n}(F)$: the set of all $n \times n$ matrices over the field F
- $\mathbb{R}$: the field of all real numbers
- $\mathbb{C}$: the field of all complex numbers
- A^*: the conjugate transpose of the matrix A

1. (12%) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the $\mathbb{R}$-linear map defined by

$$T(a,b,c) = (a-3b-2c, a+b, 3a+5b+c).$$

Find the rank of T, and find a basis for the null space of T.

2. (12%) Suppose W_1 and W_2 are the following subspaces of the real vector space $M_{3\times 3}(\mathbb{R})$:

$$W_1 = \left\{ \begin{pmatrix} a & 2a & b \\ b & c & 0 \\ 0 & 0 & d \end{pmatrix} : a,b,c,d \in \mathbb{R} \right\}, \quad W_2 = \left\{ \begin{pmatrix} a & b & 2a \\ b & 2c & d \\ 0 & d & 0 \end{pmatrix} : a,b,c,d \in \mathbb{R} \right\}.$$

Find the dimension of the subspace $W_1 + W_2$.

3. Consider the real matrix $A = \begin{pmatrix} 12 & -5 & -5 & 3 \\ 20 & -8 & -10 & 0 \\ 7 & -3 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

   (a) (10%) Find the characteristic polynomial of A.

   (b) (5%) Is A similar to a 4×4 diagonal matrix over $\mathbb{R}$? Justify your answer.

   (c) (5%) Is A similar to a 4×4 diagonal matrix over $\mathbb{C}$? Justify your answer.

4. (12%) Show that if A is a 3×3 real matrix, then A is similar to

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} 0 & \lambda & 0 \\ 1 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 0 & 0 & \lambda \\ 1 & 0 & \mu \\ 0 & 1 & \nu \end{pmatrix}$$

for some $\lambda, \mu, \nu \in \mathbb{R}$.

5. (12%) Let A be a 6×6 complex matrix such that $A^3 = 0$. Find all possible Jordan canonical forms of A.

6. (12%) Suppose $N \in M_{n\times n}(\mathbb{C})$ is normal, i.e., $N^*N = NN^*$. Show that N is self-adjoint if and only if all eigenvalues of N are real.

7. Let $\langle A, B\rangle$ be the trace of AB^* for all $A, B \in M_{n\times n}(\mathbb{C})$.

   (a) (10%) Show that $\langle \cdot, \cdot \rangle$ is an inner product on $M_{n\times n}(\mathbb{C})$.

   (b) (10%) Let $P \in M_{n\times n}(\mathbb{C})$ be invertible, and let T be the linear operator on $M_{n\times n}(\mathbb{C})$ defined by $T(A) = P^{-1}AP$. Find the adjoint of T with respect to the inner product $\langle \cdot, \cdot \rangle$.