成功大學 102 學年度「碩士班」研究生甄試入學考試

【基礎數學】: Part I. 高等微積分

Master Entrance Exam, Advanced Calculus, NCKU Math, Oct. 26, 2012

Show all works

1. Prove or disprove the following statements.

(a) If f is continuous at $x = 0$, then there exists a $\delta > 0$ such that f is continuous on $(-\delta, \delta)$. [5%]

(b) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$, then f' is continuous on $\mathbb{R}$. [5%]

(c) Let $f : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Consider a smooth surface $S = \{(x, y, z) : f(x, y, z) = k\}$ for some constant k and a curve $\{\alpha(t) = (x(t), y(t), z(t)); \alpha(t) \subset S,\ t \in I\}$ for some interval I. Show that ∇f is perpendicular to the surface S. [10%]

2. (a) State the Inverse Function Theorem. (Do not prove it.) [5%]

(b) Let $f(x) = \begin{cases} x + x^3 \sin \frac{1}{x}, & x \neq 0. \\ 0, & x = 0. \end{cases}$ Using the Inverse Function Theorem to prove that $f(x)$ has an inverse on a small neighborhood containing 0. [10%]

3. (a) Let $a_n > 0$ and $\sum_n a_n$ converge. Do the series $\sum_n a_n^2$ and $\sum_n \sqrt{a_n}$ converge? or diverge? Prove it if it is correct. Disprove it if it is wrong by giving examples. [5%]

(b) State the definitions of a function being continuous, uniformly continuous, and absolutely continuous. Also gives examples of functions being uniformly continuous but not absolutely continuous, and continuous but not uniformly continuous. [10%]

(c) Give examples for function being Riemann integrable but not Lebesgue intetgrable and being Lebesgue intetgrable but not Riemann integrable. [10%]

4. Let ϕ be continuous and bounded. Let $u_\epsilon(\mathrm{x}, t) = \dfrac{1}{(2\pi t)^{\frac{3}{2}}(\epsilon + i)^{\frac{3}{2}}} \iiint_{\mathbf{R}^3} e^{\frac{-|\mathrm{x}-\mathrm{x}'|^2}{2(\epsilon+i)t}} \phi(\mathrm{x}')\, d\mathrm{x}'$ with $\epsilon, t > 0$, where $i = \sqrt{-1}$. State the theorems you use in the following problems.

(a) Use the change of variables $\mathrm{x} - \mathrm{x}' = (2(\epsilon^2 + 1)t)^{\frac{1}{2}}\mathrm{y}$ to show that [10%]

$$\lim_{t \to 0^+} u_\epsilon(\mathrm{x}, t) = \frac{(\epsilon - i)^{\frac{3}{2}}}{\pi^{\frac{3}{2}}} \iiint_{\mathbf{R}^3} e^{-(\epsilon - i)|\mathrm{y}|^2}\, d\mathrm{y}\phi(\mathrm{x}).$$

(b) Use the spherical coordinates and integration by parts to show that [10%]

$$\lim_{t \to 0^+} u_\epsilon(\mathrm{x}, t) = \frac{2(\epsilon - i)^{\frac{1}{2}}}{\sqrt{\pi}} \int_0^\infty e^{-(\epsilon - i)r^2}\, dr\phi(\mathrm{x}).$$

(c) Use the facts $\int_0^\infty \cos r^2 dr = \int_0^\infty \sin r^2 dr = \sqrt{\frac{\pi}{8}}$, $e^{i\theta} = \cos\theta + i\sin\theta$, and $\lim_{\epsilon \to 0^+} (\epsilon - i)^{\frac{1}{2}} = \frac{1-i}{\sqrt{2}}$ to show that $\lim_{\epsilon \to 0^+} \lim_{t \to 0^+} u_\epsilon(\mathrm{x}, t) = \phi(\mathrm{x})$. [10%]

(d) Show that the integral $\int_0^\infty \cos r^2 dr$ converges. [10%]

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【基礎數學】: Part II. 線性代數

Notation

- A^T: the transpose of the matrix A
- I_n: the $n \times n$ identity matrix
- $\mathbb{R}$: the field of real numbers

1. (10%) Compute the dimension of the subspace

$$\{(a+2b+3c-d, a+c+d, a+2b+3c-d, 4a+5b+9c-d) \mid a,b,c,d \in \mathbb{R}\}$$

of $\mathbb{R}^4$.

2. Let V be the vector space of all polynomials of degree at most 3 with real coefficients. Suppose $T\colon V \to V$ is the map defined by

$$T(p(x)) = p(x) + p'(x),$$

where $p'(x)$ is the derivative of $p(x)$.

   (a) (4%) Show that T is a linear operator on V.

   (b) (8%) Find $[T]_\beta$ where $\beta = \{1, x, x^2, x^3\}$. (Here $[T]_\beta$ is the matrix representation of T with respect to the ordered basis β.)

3. (10%) Let

$$A = \begin{pmatrix} 2/7 & 3/7 & 6/7 \\ a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix}.$$

Find suitable $a_i \in \mathbb{R}$, $i = 1, \ldots, 6$, such that $A^TA = AA^T = I_3$.

4. (10%) Show that there are *no* 5×5 invertible matrices A, B over $\mathbb{R}$ satisfying $AB = -BA$.

5. Let A be an $m \times n$ matrix over $\mathbb{R}$.

   (a) (7%) Show that $\operatorname{rank}(BA) \le \operatorname{rank}(A)$ for every $n \times m$ matrix B over $\mathbb{R}$.

   (b) (7%) Show that $\operatorname{rank}(A^TA) = \operatorname{rank}(A)$.

6. Consider the following matrix

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 4 & 1 & 0 \\ -1 & 1 & 2 \end{pmatrix}.$$

   (a) (5%) Find the eigenvalues of A.

   (b) (12%) Find an invertible matrix Q such that $Q^{-1}AQ$ is a diagonal matrix.

   (c) (5%) Find a matrix B such that $B^3 = A$.

7. (a) (10%) Show that if A is a 102×102 matrix over $\mathbb{R}$ such that $A^{102} = 102A$, then the matrix $A - I_{102}$ is invertible.

   (b) (12%) Find a 102×102 matrix B over $\mathbb{R}$ such that $B^{102} + B^{101} + \ldots + B + I_{102} = 0$. Justify your answer.