成功大學 101 學年度「碩士班」研究生甄試入學考試

【基礎數學】: Part I. 高等微積分

1. (10) Let f be a real-valued function with $(n+1)$ derivatives on $[a,b]$. Assume that $f^{(i)}(a) = f^{(i)}(b) = 0, \quad i = 1, 2, \ldots, n$. Show that there exists a $\xi \in (a,b)$ such that $f^{(n+1)}(\xi) = f(\xi)$.

2. (10) Denote $\sin_0 x = x$. For $n \geq 1$, we define recursively $\sin_n x := \sin(\sin_{n-1} x)$. Show that

$$\lim_{x \to 0} \frac{\sin_n x}{x} = 1, \quad \forall n \in \mathbb{N}$$

3. (15) Given $0 < x_j < \pi$, $j = 1, 2, \ldots, n$. Write $x = \frac{x_1 + x_2 + \cdots + x_n}{n}$. Prove

$$\prod_{j=1}^{n} \frac{\sin x_j}{x_j} \leq \left(\frac{\sin x}{x}\right)^n$$

4. (15) Prove that the sequence $\{x_n\}$ converges to $x \in \mathbb{R}$ if and only if every subsequence of it has a subsequence that converges to x.

5. (15) For a continuous function $f : \mathbb{R} \to \mathbb{R}$, prove or disprove the following
   (a) (10) If $f(x) \geq 0$ for every rational x, then $f(x) \geq 0, \forall x \in \mathbb{R}$.
   (b) (5) If $f(x) > 0$ for every rational x, then $f(x) > 0, \forall x \in \mathbb{R}$.

6. (10) For what value of $a > 1$ is

$$\int_a^{a^2} \frac{1}{x} \ln \frac{x-1}{32} dx$$

minimum ?

7. (10) Compute for $n \in \mathbb{N}$, the integral

$$\int_{-\pi}^{\pi} \frac{\sin nx}{(1+2^x)\sin x} dx$$

8. (15) Consider the Riemann function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, \text{ if } x = 0 \text{ or if } x \notin \mathbb{Q} \\ \frac{1}{n}, \text{ if } x = \frac{m}{n}, m, n \in \mathbb{N}, \ (m,n) = 1 \end{cases}$$

Show by the definition of (Riemann) integrability that f is Riemann integrable and $\int_0^1 f(x)dx = 0$.

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【基礎數學】：Part II. 線性代數

(1) (25pts) The set of all polynomials of one variable x with real coefficients is denoted by $\mathbb{R}[x]$. Let

$$V = \{f \in \mathbb{R}[x] : \deg(f) \leq 4, f(-1) = f(1) = 0\}.$$

(a, 5pts) Prove that V with polynomial addition is a vector space over $\mathbb{R}$.

(b, 10pts) For $f, g \in V$, let

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx.$$

Prove that $(V, \langle \bullet, \bullet \rangle)$ is an inner product space.

(c, 10pts) Compute the dimension of V and find an orthonormal basis.

(2) (10pts) Let A be a 2×2 real matrix such that $A^T = -A$ and $\det(A) \neq 0$. Let g be a 2×2 real matrix. Prove $g^T A g = A$ if and only if $\det(g) = 1$.

(3) (15pts) A complex matrix A is called unitary if $A^T \bar{A} = I$.

(a, 8pts) Prove that if A is unitary, then A is diagonalizable.

(b, 7pts) Prove that if A is unitary, then all its eigenvalues have absolute value 1.

(4) (10pts) For A be an $n \times n$ real matrix, denote by $P_A(x)$ the characteristic polynomial of A. Let g be an invertible $n \times n$ real matrix. Prove $P_{gAg^{-1}}(x) = P_A(x)$.

(5) (20pts) Let A be an 3×3 matrix over $\mathbb{R}$ such that $A^m = 0$ for some positive integer m.

(a, 10pts) Compute the eigenvalues of A. You must show work and prove your claim.

(b, 10pts) Define $A^0 = I$. Compute the eigenvalues of e^A where $e^A = \sum_{i=0}^{\infty} \frac{A^i}{i!}$. You must show work and prove your claim.

(6) (20pts) Let $M_2(\mathbb{R})$ be the vector space of 2×2 real matrices and $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Let T be the linear transformation

$$T : M_2(\mathbb{R}) \to M_2(\mathbb{R}), \quad T(B) = AB - BA.$$

Compute the eigenvalues and eigenvectors of T.