國立成功大學九十六學年度 博士班入學考試 基礎數學試卷 全一頁

一、本卷有試題 10 題，共 100 分。	二、作答時請務必標清楚題號

Part I: Algebra

1. (10 points) Let G be a finite group and H a subgroup of G. Show that the order of G is divisible by the order of H.

2. (10%) Let G be a nonabelian group of order p^3, p a prime.
   (a) Show that the center of G is of order p, and
   (b) Show that the center of G is equal to the commutator subgroup $[G, G]$.

3. (10 %) Let D be an principal ideal domain. Show that every nonzero prime ideal of D is also a maximal ideal.

4. (10%) Let K be a finite field and F a finite extension over K. Show that F is Galois over K and the Galois group of F over is a cyclic group.

5. (10%) Let V be a finite dimensional real inner product space with the inner product $\langle\ ,\ \rangle$. For any unit vector $a \in V$, define $\rho_a : V \to V$ by $\rho_a(x) = x - 2\langle a, x\rangle a$.
   (a) Show that ρ_a is an isometry, i.e., $\langle \rho_a(x), \rho_a(y)\rangle = \langle x, y\rangle$ for any $x, y \in V$.
   (b) $\det \rho_a = -1$.

Part II: Analysis

1. (20 points)
   (a) Let f be a real-valued function on $[0,1]$ such that $|f|$ is (Lebesgue) measurable and the set $\{x \in [0,1] \mid f(x) > 0\}$ is measurable. Prove that f is measurable.
   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let C be a closed subset of $\mathbb{R}$. Prove that $f(C)$ is Lebesgue measurable.
   (c) Let (X, μ) be a finite measure space and $f \in L^1(X, d\mu)$. Show that for any $\varepsilon > 0$ there is a $\delta > 0$ such that $|\int_E f(x)d\mu(x)| < \varepsilon$ for any measurable set $E \subset X$ such that $\mu(E) < \delta$.
   (d) Let μ be a measure (on some measure space X) and let f_n, $n = 1, 2\dots$, be a sequence of real-valued measurable functions on X. Suppose that for every $\varepsilon > 0$, the sum

$$\sum_{n=1}^{\infty} \mu\{x \mid |f_n(x)| > \varepsilon\}$$

   is finite. Prove that f_n converges to zero almost everywhere.

2. (10 points) The convolution of two functions $f, g \in L^1(\mathbb{R})$ is defined by:

$$f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy.$$

   (a) Prove that the integral defining $f * g$ exists almost everywhere in x (with respect to the Lebesgue measure).
   (b) Prove that $\|f * g\|_{L^1} \le \|f\|_{L^1}\|g\|_{L^1}$.

3. (10 points) Let $I = (0, 1)$. Suppose $\{f_n\}_{n=1}^{\infty}$ is a norm-bounded sequence of functions in $L^2(I)$ that converges in measure to a function f.
   (a) Show that $f \in L^2(I)$ and $\|f\|_2 \le \liminf \|f_n\|_2$.
   (b) Show that $\|f_n\|_2$ converges to $\|f\|_2$ if and only if $\|f_n - f\|_2 \to 0$.

4. (10 points) Let λ denote Lebesgue measure and let $f : [0,1] \to [0,1]$ be a differentiable function such that for every Lebesgue measurable set $A \subset [0,1]$ one has $\lambda(f^{-1}(A)) = \lambda(A)$. Prove that either $f(x) = x$ or $f(x) = 1 - x$ (the derivative f' is not assumed to be continuous).