一、本卷有試題10題，共100分。 二、作答請務必標清楚題號。

Part I. ALGEBRA

[10points] 1. Let G be a group written additively with identity element 0. Note that G may not be abelian. For mappings $f : G \to G$ and $g : G \to G$, one can define the sum of f and g by $(f+g)(x) = f(x) + g(x)$ for all $x \in G$. A mapping f of G is said to be *nilpotent* if there is some positive integer n such that $f^n(x) = 0$ for all $x \in G$. Here f^n is f composed with itself n times. (For example $f^2(x) = (f \circ f)(x) = f(f(x))$ and $f^3(x) = (f \circ f)(f(x)) = f(f(f(x)))$ for all $x \in G$.)

Show that if f and g are nilpotent endomorphisms of G such that $f \circ g = g \circ f$, then $f + g$ is nilpotent.

[10points] 2. Let R be a ring with 1. An ideal M of R is said to be *maximal* if $M \neq R$ and for any ideal I such that $M \subseteq I \subseteq R$, either $I = M$ or $I = R$. Use Zorn's Lemma ro show that every proper ideal of R is contained in a maximal ideal.

[10points] 3. Let R be a ring. We say that R is *prime* if for any nonzero elements $a, b \in R$, there is an element $x \in R$ such that $axb \neq 0$. Show that the following statements are equivalent.

(i) R is prime.

(ii) If I and J are ideals of R such that $IJ = 0$, then either $I = 0$ or $J = 0$.

[10points] 4. Let F be a field of characteristic $p > 0$, and $a \in F$. Consider the polynomial $f(x) = x^p - a \in F[x]$. Prove that $f(x)$ is irreducible unless $a = b^p$ for some $b \in F$.

[10points] 5. Let U, V, X and Y be vector spaces over $\mathbb{F}$. Let $\alpha : X \to U$ and $\beta : Y \to V$ be linear transformations. Show that $\ker(\alpha \otimes \beta) = \ker(\alpha) \otimes Y + X \otimes \ker(\beta)$.

Part II. ANALYSIS

[15points] 6. (a) Let $f_k, k = 1, 2, \ldots$, be nonnegative and measurable on E. Prove that

$$\int_E \left(\sum_{k=1}^{\infty} f_k \right) = \sum_{k=1}^{\infty} \int_E f_k.$$

(b) Let f be nonnegative and measurable on E and assume that $\int_E f < \infty$. Prove that $f < \infty$ *a.e.* in E.

(c) Let $\{f_k\}$ e a sequence of measurable functions on E and assume that $\sum_{k=1}^{\infty} \int_E |f_k| < \infty$. Show that $\sum_{k=1}^{\infty} f_k$ converges absolutely *a.e.* in E.

[10points] 7. Let $\phi(x) \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \phi = 1$. If $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, show that $\lim_{\epsilon \to 0} \|f_\epsilon - f\|_p = 0$, where $f_\epsilon(x) = f * \phi_\epsilon(x) = \int_{\mathbb{R}^n} f(x-t)\phi_\epsilon(t)dt$, and $\phi_\epsilon(t) = \epsilon^{-n}\phi(\frac{t}{\epsilon})$ for each $\epsilon > 0$.

[10points] 8. Let f, $\{f_k\} \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, and assume that $f_k(x) \to f(x)$ *a.e.* and $\|f_k\|_p \to \|f\|_p$. Show that $\lim_{k \to \infty} \|f_k - f\|_p \to 0$.

[10points] 9. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue measurable function such that

$$m(\lambda) = \mu\big(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}\big) \leq C\lambda^{-2}, \quad \lambda > 0.$$

Prove that there is a constant C_1 such that for any Borel set $E \subset \mathbb{R}^n$ of finite and positive measure

$$\int_E |f(x)|\, d\mu(x) \leq C_1 \sqrt{\mu(E)}.$$

[5points] 10. Assume that $(X, \mathscr{A}, \mu)$ is a measure space, and $f \in L^1(X, \mathscr{A}, \mu)$ satisfies that $\|f\|_1 = \int_X |f(x)|\, d\mu(x) = 1$. Prove that

$$\int_E \log |f(x)|\, d\mu(x) \leq -\mu(E) \log \mu(E)$$

for all subsets $E \in \mathscr{A}$ with $0 < \mu(E) < \infty$.