Show all work !

Part I.

1. Let $(X, \mathcal{M}, \mu)$ be a measure space. The measure μ is called semifinite if for each $E \in \mathcal{M}$ with $\mu(E) = \infty$ there exists $F \in \mathcal{M}$ with $F \subset E$ and $0 < \mu(F) < \infty$. Show that if μ is semifinite and $\mu(E) = \infty$, then for any $c > 0$ there exists $F \subset E$ with $c < \mu(F) < \infty$. (10%)

2. For $f \in L^1_{loc}$, the Hardy–Littlewood maximal function Hf is defined by

$$Hf(x) = \sup_{r>0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y)|\, dy$$

where $B(r,x)$ is the closed ball with radius r centered at x and m is the Lebesgue measure. Show that Hf is not integrable unless $f = 0$ almost everywhere. (10%)

3. Let $\mathcal{X}$ be a normed vector space, $\mathcal{M}$ a closed subspace of $\mathcal{X}$ and $\mathcal{N}$ a finite dimensional subspace of $\mathcal{X}$. Show that $\mathcal{M} + \mathcal{N}$, which is $\{m + n : m \in \mathcal{M}, n \in \mathcal{N}\}$, is a closed in $\mathcal{X}$. (10%)

4. Let m be the Lebesgue measure. Show that $L^\infty(\mathbb{R}^n, m)$ is not separable. (10%)

5. Suppose that $1 < p < \infty$, q is the conjugate exponent to p (i.e. $p^{-1} + q^{-1} = 1$), $f \in L^p$, and $g \in L^q$. Show that $f * g \in C_0(\mathbb{R}^n)$. Recall that $f \in C_0(\mathbb{R}^n)$ if the set $\{x : |f(x)| > \epsilon\}$ is compact for every $\epsilon > 0$. (10%)

Part II.

6. Find all normal subgroups of dihedral group D_n of degree $n \geq 3$. (10%)

7. (a) If D is an integral domain contained in an integral domain E and $f \in D[x]$ has degree n, then f has at most n distinct roots in E. (4%)

(b) Given an example shows that (a) may be false without the hypothesis of commutativity. (3%)

(c) Given an example shows that (a) may be false if E has a zero divisors. (3%)

8. Let F_n be a cyclotomic extension of $\mathbb{Q}$ of order n. Determine $Aut_{\mathbb{Q}} F_5$ and all intermediate fields. (10%)

9. If $\phi\colon \mathbb{Q}^3 \to \mathbb{Q}^3$ is a linear transformation and relative to some basis the matrix of ϕ is $A = \begin{pmatrix} 0 & 4 & 2 \\ -1 & -4 & -1 \\ 0 & 0 & -2 \end{pmatrix}$. Find the invariant factors of A and ϕ and their minimal polynomial. (10%)

10. Suppose R is a commutative ring and N is the intersection of all prime ideals of R. Show that $x \in N$ if and only if x is nilpotent. (**Hint**: If x is not nilpotent, consider the family of ideals I so that $x^n \notin I$ for all $n > 0$. Apply Zorn's lemma). (10%)