88.06.03
AM 9:00-12:00

Part I.

1. Let G be a group and suppose that $|G| = 110$. Show that:
   (a) A Sylow 11-subgroup P_{11} of G is normal. (5 pt)
   (b) Find the order $|H|$ of the factor group $H = G/P_{11}$ and show that a Sylow 5-subgroup P_5 of H is normal. (6 pt)
   (c) Find the order $|H/P_5|$. Is the group G soluble? (9 pt)

2. Let $A = C[0,1]$ be the ring of all continuous real valued functions on $[0,1]$. Set $I = \{f \in A \mid f(0) = 0\}$. Show that I is a maximal ideal of A and show that the factor ring A/I is isomorphic to the real number field. (7 pt)

3. Let Z be the ring of integers. Consider the factor module $Z_{12} = Z/12Z$. Find all endomorphisms and all automorphisms of the Z-module Z_{12}. (10 pt)

4. Let F be a finite dimensional Galois extension of a field K with Galois group G. Suppose that G is a cyclic group of order 15.
   (a) Find $[F : K]$. (4 pt)
   (b) Find the number of fields E such that $K \subseteq E \subseteq F$. (9 pt)

Part II.

5. (a) Please State the Fundamental Theorem of Calculus. (4 pt)
   (b) Please State the Hahn-Banach Theorem. (3 pt)
   (c) Please State the Ascoli-Arzela Theorem. (3 pt)

6. (a) Construct a nowhere dense set with measure $\frac{1}{2}$. (5 pt)
   (b) Give two examples that a function is Riemann integrable but not Lebesgue integrable and a function is Lebesgue integrable but not Riemann integrable. (5 pt)

7. Let g be a bounded measurable function. Show that for each $\epsilon > 0$ there is an integrable function f such that

$$\int fg \geq (\|g\|_\infty - \epsilon)\|f\|_1. \qquad (10\text{ pt})$$

8. (a) Show that an open subset of a locally compact Hausdorff space is locally compact. (5 pt)
   (b) Show that in R^n "a set is closed and bounded" is equivalent to "a set is compact". (5 pt)

9. Let S be a subspace of $L^2[0,1]$, and suppose that there is a constant K such that $|f(x)| \leq K\|f\|$ for all $x \in [0,1]$. Then the dimension of S is at most K^2. (10 pt)