1. Let A be an $m \times n$ matrix. Prove that
   (a) The linear system $Ax = b$ is solvable if and only if b belongs to the column space of A. (4%)
   (b) The following statements are equivalent: (10%)
      (i) the solution of $Ax = b$ is unique;
      (ii) $Ax = 0$ has no non-trivial solution;
      (iii) rank $A = n$.

2. Let A be an $n \times n$ matrix. Prove that A is positive semi-definite if and only if there exists an $n \times n$ matrix B such that $A = B^*B$. (10%)

3. Let T be a linear operator on V, $\dim V = n < \infty$.
   (a) Let W be a T invariant subspace of V (i.e. $T(W) \subseteq W$).
   If $\mathcal{B}' = \{v_1, v_2, \ldots, v_k\}$ is a basis for W and $\mathcal{B} = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ is a basis for V.
   Find the relation of $[T]_{\mathcal{B}}$ and $[T|_W]_{\mathcal{B}'}$. (5%)
   (b) Show that the characteristic polynomial of $T|_W$ divides the characteristic polynomial of T. (5%)
   (c) For all $x \in V$, let W_x be the smallest T-invariant subspace containing x. Show that $\{x, T(x), \ldots, T^{k-1}(x)\}$ is a basis of W_x for some integer k. (5%)
   (d) Let $\mathcal{B}' = \{x, T(x), \ldots, T^{k-1}(x)\}$ (as (c)). Find $[T|_{W_x}]_{\mathcal{B}'}$. (5%)
   (e) State and prove the Caley-Hamilton theorem. (3%)

4. Suppose f and g are continuous functions on $[a, b]$.
   (a) Show that if $g(x) \geq 0$ for all $x \in [a, b]$, then there exists $c \in [a, b]$ such that $\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$. (8%)
   (b) Show that the conclusion in (a) is false when the condition "$g(x) \geq 0$" is dropped. (5%)

5. Let $f_n : [0, 1] \to \mathbb{R}$ be defined by $f_n(x) = \dfrac{x}{(1+x)^n}$ for $n = 0, 1, 2, \ldots$.
   (a) Prove that $\sum_{n=0}^{\infty} f_n(x)$ is convergent for all $x \in [0, 1]$. (4%)
   (b) Is it uniformly conergent on $[0, 1]$? Justify your answer. (5%)
   (c) Does $\int_0^1 \sum_{n=0}^{\infty} f_n(x)\,dx = \sum_{n=0}^{\infty} \int_0^1 f_n(x)\,dx$? (6%)

6. Evaluate the integral $\iint_D \sin\left(\frac{y-x}{x+y}\right) dA$ where $D = \{(x, y) \mid 0 < x < y < 1 - x\}$. (10%)

7. Let m be the Lebesgue measure on $\mathbb{R}$, $f \in L^1(\mathbb{R})$ and define $g(x) = \int_x^{x+1} f(t)\,dt$. Show that g is a continuous function and $\lim_{x \to \infty} g(x) = 0$. (10%)

(題目請用本張複印製版請以黑色正楷書寫或打字並請勿超出此線)

命題人簽章