General Analysis

1. (a) Let $\mathcal{P} = \{A_t\}_{t \in T}$ be a partition of a set Ω, i.e., a family of pairwise disjoint nonempty subsets of Ω with union Ω. Describe the σ-field of subsets of Ω generated by $\mathcal{P}$. (5%)

   (b) Use (a) to show that no partition of the real line $\mathbb{R}$ can generate the σ-field $\mathcal{B}$ of Borel subsets of $\mathbb{R}$. (5%)

2. Let X be a real-valued measurable function defined on a measure space $(\Omega, \mathcal{A}, \mu)$ such that

$$\int_\Omega |X|^n \, d\mu \leq k < +\infty \qquad \text{for all } n = 1, 2, 3, \ldots$$

   where k is a finite constant.

   (a) If c is a real number larger than 1, show that

$$\mu\{|X| \geq c\} = 0.$$

   (8%)

   (b) Show that $|X| \leq 1$ μ-a.e. (8%)

3. Show that $\lim_{n\to\infty} \int_1^n (1 - \frac{t}{n})^n \log t \, dt = \int_1^\infty e^{-t} \log t \, dt$. (8%)

4. Let μ be a finite measure defined on the σ-algebra of Borel subsets of the real line $\mathbb{R}$. Determine whether or not the following limits exist? If so, say why, and find the limit. If not, say why not.

   (a) $\lim_{n\to\infty} \int_{\mathbb{R}} (1 - \cos \frac{x}{n}) \sin nx \, d\mu(x)$. (8%)

   (b) $\lim_{n\to\infty} \int_{\mathbb{R}} e^{-nx^2} \, d\mu(x)$. (8%)

5. Let μ and ν be counting measures on X and Y respectively where $X = Y = \{1, 2, 3, \ldots\}$. Suppose

$$f(x,y) = \begin{cases} 2 - 2^{-x}, & \text{if } x = y \\ -2 + 2^{-x}, & \text{if } x = y + 1 \\ 0, & \text{otherwise} \end{cases}$$

   (a) Compute two iterated integrals and show they are unequal. (10%)

   (b) Does (a) contradict Fubini Theorem? Why? (5%)

6. Let $\Omega = [0,1]$, $\mathcal{A}$ = the Borel sets of Ω,
   $\varphi \equiv \lambda$ be the Lebesgue measure on $[0,1]$,
   $\mu \equiv$ counting measure.

   (a) Show that every measure on $[0,1]$ is always absolutely continuous with respect to μ. (5%)

   (b) Show that it is impossible to represent λ as the indefinite integral with respect to μ. Namely, there is no X, finite, integrable, such that

$$\varphi(B) = \int_B X \, d\mu \qquad \text{for } B \in \mathcal{A}.$$

   (5%)

   (c) Does this example contradict the Random-Nikodym Theorem? Why? (5%)

7. Consider the $L^p(\mathbb{R}, \mathcal{B}, \lambda)$ spaces for $p \in (0, \infty)$ where $\mathcal{B}$ contains all the Borel subsets of $\mathbb{R}$ and λ is the Lebesgue measure. Construct an example showing $f_n \longrightarrow f$ uniformly on $\mathbb{R}$, however $f_n \not\longrightarrow f$ in L^p for any $p \in (0, \infty)$. (10%)

8. Let X be an integrable function on a measure space $(\Omega, \mathcal{A}, \mu)$ with $\int_\Omega X \, d\mu < 0$. Let $t \neq 0$ be a real number. If $\int_\Omega e^{tX} \, d\mu = 1$, show that $t > 0$. (10%)

Functional Analysis

1. If X is an infinite–dimensional topological vector space which is the union of countable many finite–dimensional subspaces, prove that X is of the first category in itself. Prove that therefore no infinite–dimensional F–space has a countable Hamel basis (note that a Hamel basis is a maximal linearly independent subset). (10%)

2. Let X and Y be Banach spaces, and let T be a bounded linear transformation of X into Y such that the range $\mathcal{R}(T)$ is closed in Y. Show that there exists a constant $M > 0$ such that every y in $\mathcal{R}(T)$ can be written as $y = Tx$ with $\|x\| \le M\|y\|$. In particular, if T is one - to - one, then T ise bounded below by $\frac{1}{M}$. (12%)

3. Let $X = (C[0,1], \|\cdot\|_\infty)$ and $k(s,t)$ be a continuous function on $[0,1] \times [0,1]$. Define $K : X \to X$ by $(Kx)(s) = \int_0^1 k(s,t)x(t)\,dt$.
   (a) Show that K is a bounded linear operator on X and find $\|K\|$. (8%)
   (b) Prove that K is a compact operator. (10%)

4. Let X be a normed space.
   (a) Show that if X^* (the dual space of X) is separable, then X itself is separable. (10%)
   (b) Show by an example that the separabitity of X does not imply that of X^*. (5%)

5. Suppose H is an infinite–dimensional Hilbert space.
   (a) Show that every orthonormal sequence (e_n) in H converges to 0 weakly. (6%)
   (b) Show that the closed unit ball B_1 in H is the weak closure of the unit sphere $S_1 = \{x \in H : \|x\| = 1\}$. (6%)
   (c) Construct a weakly dense subset A of H such that $A \cap B_1$ is not weakly dense in B_1. (6%)

6. Let H be a complex Hilbert space and T be a bounded operator on H with the spectrum $\sigma(T)$.
   (a) Show that $\sigma(T)$ is bounded. (5%)
   (b) Show that $\sigma(T)$ is closed. (6%)
   (c) Show that T is normal if and only if $\|Tx\| = \|T^*x\|$ for all $x \in h$. (6%)

7. Prove that the closed unit ball of $L^1[0,1]$ (relative the Lebesgue measure) has no exteme points. (10%)

1. (20 points) Let k be an algebraically closed field and let V be a vector space over k. Let V^* denote its dual space. Suppose $\{\lambda_1, \cdots, \lambda_n\}$ are n distinct elements of V^*. Prove that there exists an element $v \in V$ such that $\lambda_1(v), \cdots, \lambda_n(v)$ are all distinct elements of k.

2. (20 points) Let G be a finite group and k be a field of characteristic 0. Let $k[G]$ denote its group ring, i.e. $k[G] = \{\sum_{g\in G} a_g g | a_g \in k\}$. $k[G]$ is a vector space over k with basis $\{g | g \in G\}$ and hence has dimension $|G|$. Let V be a $k[G]$-module and let W be a submodule of V. Prove that there exists a submodule $\bar{W}$ such that $V = W \oplus \bar{W}$. (*Hint:* Let $V = W \oplus W'$ be a direct sum of vector spaces. Let $\pi_0 : V \to W$ be the projection onto W. Define $\pi : V \to W$ by $\pi(x) := \frac{1}{|G|}\sum_{g\in G} g^{-1}\pi_0(gx)$. What can this "nice" π do for you?) *You may NOT assume that $k[G]$ is semisimple!*

3. (20 points) Let k be field and $A = k[[x]]$ be the ring of formal power series in k, i.e. $k[[x]] = \{\sum_{i=0}^{\infty} a_i x^i | a_i \in k\}$. What are the units in A? Show that A has a unique maximal ideal M and every other ideal is of the form M^n, where n is a non-negative integer.

4. (20 points) Let K be a separable algebraic filed extension of a field k. Suppose that there exists a fixed integer n such that $[k[x] : k] \leq n$ for all $x \in K$. Prove that K is a finite field extension of k. (*Hint:* You may use the Primitive Element Theorem, because you have a separable extension.)

5. (20 points) Let G be a non-abelian group of order p^3, where p is a prime. Let $Z(G)$ denote its center. Let $\mathbb{Z}_p$ denote the cyclic group of order p.

a. Show that $Z(G)$ is cyclic of order p.

b. Show that $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

c. Let H be a subgroup of order p^2. Show that H contains $Z(G)$ and H is a normal subgroup.

Mathematical Statistics

1. (a) For the location family with density $f(x-\theta)$ (x, θ real-valued).

   Show that the amount of information about θ, $I(\theta)$, is independent of θ and given by

$$I_f = \int_{-\infty}^{\infty} \frac{[f'(x)]^2}{f(x)}\,dx.$$

(10%)

   (b) What is the amount of information about θ for the location–scale family with density $\frac{1}{b}f(\frac{x-\theta}{b})$? ($b > 0$) (10%)

2. Consider a random sample of size $n = 5$ for testing $H_0 : X \sim N(0,1)$ against the alternative that X has this Cauchy p.d.f.:

$$f_1(x) = \frac{1}{\pi}\cdot\frac{1}{1+x^2}, \quad x \in \mathbb{R}.$$

   Find the Neyman–Pearson rejection regions. (20%)

3. Let $X_1, \ldots, X_n$ be a random sample from a population with density $p(x, \underset{\sim}{\theta})$ given by

$$p(x; \underset{\sim}{\theta}) = \frac{1}{\sigma}\exp\left\{-\frac{(x-\mu)}{\sigma}\right\} \quad \text{if } x \geq \mu,$$

   where $\underset{\sim}{\theta} = (\mu, \sigma)$ with $-\infty < \mu < +\infty$, $\sigma > 0$.

   (a) Find the UMVU estimator of $P_{\underset{\sim}{\theta}}(X_1 \geq t)$ for $t \geq \mu$. (10%)

   (b) What is the MLE of $P_{\underset{\sim}{\theta}}(X_1 \geq t)$? (10%)

4. (a) Show that if $\sqrt{n}(T_n - \theta) \xrightarrow{\mathcal{L}} N(0, \tau^2)$, then

$$\sqrt{n}[f(T_n) - f(\theta)] \xrightarrow{\mathcal{L}} N(0, \tau^2(f'(\theta))^2),$$

   provided $f'(\theta)$ exists and is not zero. (10%)

   (b) Let $X_1 \ldots, X_n$ be i.i.d. $N(\theta, \sigma^2)$, both θ and σ^2 are unknown and let the estimand be θ^2. Find the UMVU estimator and its limiting distribution. (10%)

5. Suppose that $\wedge$ is a distribution of Θ such that

$$\int R(\theta, \delta_\wedge)\,d\wedge(\theta) = \sup_\theta R(\theta, \delta_\wedge).$$

   Show that

   (a) $\delta_\wedge$ is minimax. (10%)

   (b) $\wedge$ is least favorable. (10%)

   Note. A prior distribution $\wedge$ is said to be least favorable if $r_\wedge \geq r_{\wedge'}$ for all prior distributions $\wedge'$. Here $r_\wedge = \int R(\theta, \delta_\wedge)\,d\wedge(\theta)$.