國立成功大學 102 學年度「博士班」研究生招生入學考試

【高等微積分】

advanced calculus

1. Let the sequence $\{a_n\}$ be given recursively by the formula

$$\begin{cases} a_1 = 0, \\ a_{n+1} = \frac{1}{4} + a_n^2. \end{cases}$$

Show that $\{a_n\}$ is monotone increase and bounded above. (20 points)

2. Show that the function

$$f(x) = \int_0^{1/x} \frac{1}{1+t^4} dt - \int_x^0 \frac{t^2}{1+t^4} dt$$

is constant for $x > 0$. (20 points)

3. Evaluate the double integral $\int_0^2 \int_y^2 e^{x^2} dx dy$. (20 points)

4. For $x \in \mathbb{R}^3$, let $\rho(x)$ be a charge density that is continuous and such that $\rho(x) = 0$ for $\|x\|_2 \geq 1$. Show that the electrostatic potential, given by

$$\phi(x) = \frac{1}{4\pi} \int \int \int_{\mathbb{R}^3} \rho(y)/\|x - y\|_2 dy,$$

is a convergent integral for each $x \in \mathbb{R}^3$. (20 points)

5. Evaluate $\int_{\mathbf{R}^n} \|x\|^2 \cdot e^{-\|x\|^2} dx$. (20 points)

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【線性代數】

Linear Algebra PhD Entrance Exam Date: Friday 03/05/2013

Work out all problems and no credit will be given for an answer without reasoning.

1. (a) (5%) If V is a vector space over F of dimension 5 and U and W are subspaces of V of dimension 3, prove that $U \cap W \neq \{0\}$. Generalize.

   (b) (5%) Let $V = \mathbb{R}^3$ and $W = \{(a, b, c) \in V \mid a + b = c\}$. Is W a subspace of V? If so, what is its dimension?

   (c) (10%) Let V and W are vector spaces over a field F. Define a **vector space isomorphism** is a one-to-one linear transformation from V onto W. If V is a vector space over F of dimension n, prove that V is isomorphic as a vector space to $F^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in F\}$.

2. (a) (10%) Show that the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T([x_1, x_2, x_3]) = [x_1 - 2x_2 + x_3, x_2 - x_3, 2x_2 - 3x_3]$$

   is invertible, and find a formula for its inverse.

   (b) (10%) Let V be the vector space of 2 by 2 matrices over $\mathbb{R}$ and let

$$M = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}.$$

   Let $T : V \to V$ be the linear transformation defined by $T(A) = AM - MA$. Find a basis and the dimension of the kernel W of T.

3. (a) (5%) Prove that if A is a square matrix, then AA^T and A^TA have the same eigenvalues.

   (b) (8%) Diagonalize the matrix

$$A = \begin{bmatrix} -3 & 5 \\ -2 & 4 \end{bmatrix},$$

   and compute A^k in terms of k.

   (c) (7%) Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{bmatrix}$$

   Find $\det(A - I_5)$.

4. (a) (10%) Find the minimal polynomial $m(t)$ of

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix}.$$

(b) (10%) Find a Jordan canonical form and a Jordan basis for the given matrix:

$$A = \begin{bmatrix} -3 & 0 & 1 \\ 2 & -2 & 1 \\ -1 & 0 & -1 \end{bmatrix}.$$

5. (a) (10%) Find an orthogonal basis for the subspace spanned by the set $\{1, \sqrt{x}, x\}$ of the vector space $C_{[0,1]}$ of continuous functions with domain $0 \leq x \leq 1$, where the inner product is defined by $\langle f, g \rangle = \int_0^1 f(x)g(x)\, dx$.

(b) (10%) Let T be a linear operator on a finite dimensional inner product space V. Show that there exists a unique linear operator T^* on V such that

$$\langle T(u), v \rangle = \langle u, T^*(v) \rangle$$

for every $u, v \in V$.

【實變數函數論】

Real Analysis

1. **Lebesgue outer measure in $\mathbb{R}$**
   (a) (5) State the definition of the Lebesgue outer measure.
   (b) (25) Show that the Lebesgue outer measure of an interval is its length.
2. **Lebesgue measurable real-valued function**
   (a) (5) State the definition of a Lebesgue measuable real-valued function.
   (b) (25) Show that the pointwise a.e. limit of a sequence of Lebesgue measuable real-valued functions is again measurable.
3. **Lebesgue integral in $\mathbb{R}$**
   (a) (5) State the definition of the Lebesgue integral of a nonnegative measurable function.
   (b) (35) state and prove **Fatou's Lemma**.

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【代數】

Algebra Exam

May 2013

$\mathbb{Z}$ = integers. $\mathbb{Q}$ = rational numbers. $\mathbb{C}$ = complex numbers.

1. (20 points) Determine whether each statement below is true or false. If true, prove the statement. If false, provide a counterexample.
   (a) Every prime ideal of every commutative ring with unity is a maximal ideal.
   (b) If D is an integral domain, then $D[x]$ is an integral domain.
   (c) Let $\mathbb{F}$ be a field. Every principal ideal of $\mathbb{F}[x]$ is a maximal ideal.
   (d) A ring homomorphism $\phi : R \to R'$ carries ideals of R into ideals of R'.
2. (10 points) Let H and K be subgroups of a group G. Show that the index of $H \cap K$ in H is at most equal to the index of K in G:
$$[H : H \cap K] \leq [G : K].$$
3. Let G be a simple group of order 60.
   (a) (8 points) Show that G has six Sylow 5-subgroups.
   (b) (8 points) Show that G has ten Sylow 3-subgroups.
4. (10 points) Let $p \neq q$ be prime numbers. Prove that no group of order $p^2 q$ is simple.
5. (10 points) Determine the Galois group of $p(x) = x^5 - 4x + 2$ over $\mathbb{Q}$.
6. What is the Galois group of $p(x) = x^3 - x + 4$, considered over the ground fields
   (a) (5 points) $\mathbb{Z}_3$,
   (b) (5 points) $\mathbb{R}$,
   (c) (8 points) $\mathbb{Q}$.
7. Let X be a topological space. Consider the ring $R(X)$ of continuous real-valued functions on X. The ring structure is given by point-wise addition and multiplication.
   (a) (8 points) Show that for each $x \in X$ the set
$$M_x = \{f \in R(X) \mid f(x) = 0\}$$
   is a maximal ideal in $R(X)$.
   (b) (8 points) Show that if X is compact, that is, every open covers of X has a finite subcover, then every maximal ideal in $R(X)$ is equal to M_x for some $x \in X$.

This exam has 7 questions, for a total of 100 points.

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【微分幾何】

1. (20 pts) Let S be the elliptic paraboloid given by

$$S = \{(x, y, z) \in \mathbb{R}^3 |\ 2z = x^2 + y^2\}.$$

Find the Gauss curvature $K(x, y, z)$ and the mean curvature $H(x, y, z)$ of S.

2. (25 pts) Let $\alpha : I \to \mathbb{R}^3$ be a regular curve parametrized by arc length s with curvature $k(s)$ and torsion $\tau(s)$. Assume that $\tau(s) \neq 0$ and $k'(s) \neq 0$ for all $s \in I$. Show that a necessary and sufficient condition for $\alpha(I)$ to lie on a sphere is that

$$R^2 + (R')^2 T^2 = \text{ const.},$$

where $R = 1/k$, $T = 1/\tau$, and R' is the derivative of R relative to s.

3. (15 pts) Let S be a regular orientable surface. Show that the mean curvature H at $p \in S$ is given by

$$H = \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\theta)\, d\theta,$$

where $\kappa_n(\theta)$ is the normal curvature at p along a direction making an angle θ with a fixed direction.

4. (15 pts) Let S be a connected and orientable regular surface. Prove that $H^2 \geq K$ where K and H are the Gauss curvature and mean curvature of S, respectively. If $H^2 = K$ for all $p \in S$, what is S?

5. (25 pts) Show that the circular cylinder $S^1 \times (0, 1)$ is a regular surface.