注意事項：作答時請務必在所屬答案卷上作答並標明題號。 101.05.08

Advanced Calculus

1. **Bolzano-Weierstrass Theorem**
   (a) (15) Show the $\mathbb{R}^n$ version of this theorem.
   (b) (10) Show that the assumption " $\mathbb{R}^n$ " is essential.
2. **Norms**
   (a) (15) Show that all norms in $\mathbb{R}^n$ are equivalent.
   (b) (10) Show that the assumption " $\mathbb{R}^n$ " is essential.
3. **Compactness**
   (25) Let S be a subset of a metric space. Then S is compact if and only if every sequence in S contains a convergent subsequence in S.
4. **Series**
   (a) (15) If the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then every rearrangement of it also converges to the same value.
   (b) (10) Show that the assumption of absolute convergence is essential.

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Work out all of the following problems with details.

16 Pts 1. Let V and W be finite dimensional vector spaces over the field F. Let $L : V \to W$ be a linear map. Prove that the dimension of the kernel of L plus the dimension of the image of L is equal to the dimension of V.

16 Pts 2. In each of the following cases decide whether there is a linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that the following holds:

(a) $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$, and $T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$?

(b) $T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$?

In case such a linear map T exists, determine its matrix with respect to the standard basis of $\mathbb{R}^2$ and with respect to the standard basis of $\mathbb{R}^3$. If no such a linear map T exists, explain why it is so.

20 Pts 3. Let F be a field and V the vector space F^2. Let $T : V \to V$ be a linear operator. A vector $\alpha \in V$ is said to be a *cyclic vector* for T if $\{T^i\alpha \mid i = 0, 1, 2 \dots\}$ spans V.

(a) Prove that any nonzero vector of V which is not a eigenvector for T is a cyclic vector for T.

(b) Prove that either T has a cyclic vector or T is a scalar multiple of the identity operator.

16 Pts 4. Let S be a subspace of a finite dimensional inner product space V over either $\mathbb{R}$ or $\mathbb{C}$. Prove that each coset in V/S contains exactly one vector that is orthogonal to S.

16 Pts 5. Let M be an $n \times n$ with real entries, $n \geq 1$. Suppose that M is unitary, upper triangular, and has positive entries on the main diagonal. Prove that M is the identity matrix

16 Pts 6. A square matrix N over a field is said to be *nilpotent* if $N^k = 1$ for some $k \geq 0$. Let N_1 and N_2 be 3×3 nilpotent matrices over the field F. Prove that N_1 and N_2 are similar if and only if they have the same minimal polynomial.

Total number of points: 100

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Real Analysis PhD Entrance Exam May 8, 2012

1. (8 points) Compute the limit

$$\lim_{n\to\infty}\int_0^\infty \frac{n\cos x}{1+n^2x^{3/2}}dx.$$

2. (12 points) For $f\in L^p(0,\infty)$, $1\le p\le\infty$, define

$$(Tf)(y)=\int_0^\infty (x+y)^2 e^{-(x+y)}f(x)\,dx \quad \text{for } y\in(0,\infty).$$

Show that $Tf\in L^p(0,\infty)$ and $\|Tf\|_p\le 2\|f\|_p$.

3. (10 points) Suppose μ is a positive measure on X and $f: X\to(0,\infty)$ satisfies $\int_X f d\mu=1$. Prove, for every $E\subset X$ with $0<\mu(E)<\infty$, that

$$\int_E(\log f)\ d\mu\le\mu(E)\log\frac{1}{\mu(E)}$$

and, when $0<p<1$,

$$\int_E f^p d\mu\le\mu(E)^{1-p}.$$

4. (10 points) Suppose $E\subseteq\mathbb{R}$ is measurable with $|E|=\lambda>0$, where λ is a finite number. Show that for any $0<t<\lambda$, there exists a subset A of E such that A is measurable and $|A|=t$. That is, the Lebesgue measure $|\cdot|$ on $\mathbb{R}$ satisfies the Intermediate Value Theorem.

5. (10 points) Let f_k and f be (Lebesgue) measurable on a measurable set $E\subset\mathbb{R}^n$, $|E|<\infty$. Then

$$f_k\to f \quad\text{in measure}\quad \text{iff}\quad \int_E\frac{|f_k-f|}{1+|f_k-f|}dx\to 0\quad \text{as } k\to\infty.$$

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Algebra Exam May 2012

Show ALL work for full credit.

(1) (10pts) Let $p < q$ be primes with $q \not\equiv p \pmod{p}$.
(a) Show that every group of order pq is cyclic.
(b) Let G be a group and $H < Z(G)$, where $Z(G)$ denotes the center of G. Suppose G/H is cyclic. Prove that G is abelian.

(2) (10pts) Let R be a commutative ring with identity. Let $N = \{r \in R : r^n = 0 \text{ for some } n > 0\}$.
(a) Prove N is an ideal.
(b) Suppose N is a maximal ideal of R. Prove that N is the unique maximal ideal of R.

(3) (10pts) Let R be a commutative ring with unity. Suppose the following diagram R-modules commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

and the rows are exact.
(a) Prove that if f and h are surjective, then g is surjective.
(b) Prove that if f and h are injective, then g is injective.

(4) (10pts) Let K be the splitting field of $x^3 - 2$ over $\mathbb{Q}$. Compute the Galois group of K over $\mathbb{Q}$ and all the intermediate fields.

(5) (10pts) Let $\mathbb{F}$ be a finite field.
(a) Prove that $|\mathbb{F}| = p^r$ where $p, r \in \mathbb{Z}_+$ with p a prime.
(b) Let $p \in \mathbb{Z}$ be a prime and $\mathbb{F}_p$ the finite field of p elements. Let $\mathbb{F}$ be an extension field of $\mathbb{F}_p$. Prove that the Galois group $Gal(\mathbb{F}/\mathbb{F}_p)$ is cyclic. (Hint: Consider the Frobenius isomorphism $x \mapsto x^p$)