Calculus Midterm #2 (Form D)

(1) Yes. For a concave-up curve, it lies above its tangent line. Conversely, if the curve is concave down, it lies below its tangent line. Since a function changes the concavity at points of inflection, the tangent line definitely cross the graph of the function. ∎

(2)

$$f(x) = \sqrt{|x-1|} = \begin{cases} \sqrt{x-1}\,, & \text{if } x \geq 1; \\ \sqrt{1-x}\,, & \text{if } x < 1. \end{cases}$$

(i) First, $f(1) = 0$ is defined. For the limit at $x = 1$, we check

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \sqrt{x-1} = 0,$$

and

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \sqrt{1-x} = 0,$$

which implies

$$\lim_{x \to 1} f(x) = 0 = f(1).$$

So, f is continuous at $x = 1$.

(ii) By either

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x-1} = \lim_{x \to 1^+} \frac{\sqrt{x-1} - 0}{x-1} = \lim_{x \to 1^+} \frac{1}{\sqrt{x-1}} = \infty$$

or

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x-1} = \lim_{x \to 1^-} \frac{\sqrt{1-x} - 0}{x-1} = \lim_{x \to 1^-} \frac{-1}{\sqrt{1-x}} = -\infty.$$

We know f is *not* differentiable at $x = 1$. ∎

(3) Find the slope of the tangent at $(-2, 2)$ of a curve $y^2 = \frac{20-x^2}{2x} + 8$.
Sol. By implicit differentiation,

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(\frac{20-x^2}{2x} + 8).$$

Hence,

$$2yy' = \frac{-2x(2x) - (20-x^2) \cdot 2}{(2x)^2} + 0.$$

Plug in $x = -2$ and $y = 2$ to obtain $4y' = \frac{-16-32}{16} = -3$, i.e., $y' = -\frac{3}{4}$. Therefore, the slope of the curve is $-\frac{3}{4}$. ∎

(4)

(i) We have

$$\frac{dp}{dx} = \frac{1}{3}(9-x^3)^{-\frac{2}{3}}(-3x^2) = -x^2(9-x^3)^{-\frac{2}{3}}.$$

Then,

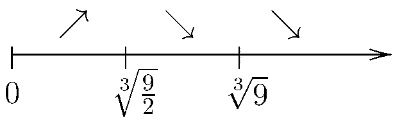
$$\eta = \frac{p/x}{dp/dx} = \frac{(9-x^3)^{\frac{1}{3}}x^{-1}}{-x^2(9-x^3)^{-\frac{2}{3}}} = -\frac{9-x^3}{x^3} = \frac{x^3-9}{x^3}.$$

Let $x = 1$, then $|\eta| = |\frac{-8}{1}| = 8 > 1$. Therefore, the demand is elastic. For an economic interpretation, a 1% decrease in price results in an 8% increase in the demand quantity at $x = 1$.

(ii) The total revenue $R = px = x(9-x^3)^{\frac{1}{3}}$. Hence,

$$\begin{aligned} R' &= (9-x^3)^{\frac{1}{3}} + x \cdot \frac{1}{3}(9-x^3)^{-\frac{2}{3}}(-3x^2) \\ &= (9-x^3)^{\frac{1}{3}} - x^3(9-x^3)^{-\frac{2}{3}} \\ &= (9-x^3)^{\frac{-2}{3}}[(9-x^3)-x^3] \\ &= \frac{9-2x^3}{(9-x^3)^{\frac{2}{3}}} \end{aligned}$$

Consider $x = \sqrt[3]{9}$ and $x = \sqrt[3]{\frac{9}{2}}$, for x-values in the interval $(0, \sqrt[3]{\frac{9}{2}})$, $R'(x) > 0$; for x-values in the interval $(\sqrt[3]{\frac{9}{2}}, \sqrt[3]{9})$, $R'(x) < 0$; for x-values in the interval $(\sqrt[3]{9}, \infty)$, $R'(x) < 0$. That is,



Therefore, $x^* = \sqrt[3]{\frac{9}{2}}$, we obtain a maximum total revenue. Then $p^* = \sqrt[3]{9-x^{*3}} = \sqrt[3]{\frac{9}{2}}$. So, $(x^*, p^*) = (\sqrt[3]{\frac{9}{2}}, \sqrt[3]{\frac{9}{2}})$.

(iii) $x^* = \sqrt[3]{\frac{9}{2}}$, then

$$|\eta| = |\frac{(\sqrt[3]{\frac{9}{2}})^3 - 9}{(\sqrt[3]{\frac{9}{2}})^3}| = |-1| = 1.$$

So, the demand at x^* is of unit elastic. Let $|\eta| < 1$, then $|\frac{x^3-9}{x^3}| < 1$. Since $\eta = \frac{p/x}{dp/dx} = \frac{x^3-9}{x^3} < 0$, we need to solve $-\frac{x^3-9}{x^3} < 1$, then $x^3 - 9 > -x^3$, which gives $x > \sqrt[3]{\frac{9}{2}} = x^*$. Therefore, for x-values in the interval $(x^*, 3)$, the demand is inelastic and by (ii) the total revenue is decreasing.

(5)

(i) We have

$$C = C(t) = \frac{3t}{27+t^3}.$$

Hence,

$$\Delta C = C(2) - C(1.5) = \frac{3 \cdot 2}{27+2^3} - \frac{3 \cdot 1.5}{27+1.5^3} \approx 0.0233.$$

(ii)

$$\frac{dC}{dt} = \frac{3(27+t^3) - 3t(3t^2)}{(27+t^3)^2} = \frac{-6t^3+81}{(27+t^3)^2}.$$

Then,

$$dC = [\frac{-6t^3+81}{(27+t^3)^2}]dt.$$

Let $t = 1.5$ and $dt = 0.5$. Then,

$$dC = (\frac{-6 \cdot 1.5^3 + 81}{(27+1.5^3)^2}) \cdot 0.5 \approx 0.0329.$$

∎

(6)

(i)

$$f'(x) = \frac{-2x}{(x^2+1)^2}.$$

We obtain the critical number $x = 0$. For $x \in (-\infty, 0)$, $f'(x) > 0$; for $x \in (0, \infty)$, $f'(x) < 0$. That is,

$$\begin{array}{ccc} \nearrow & & \searrow \\ \hline & 0 & \end{array}$$

Therefore, when $x = 0$, we obtain relative maximum $f(0) = 1$.

$$\begin{aligned} f''(x) &= \frac{-2(x^2+1)^2 - (-2x)\cdot 2(x^2+1)\cdot 2x}{(x^2+1)^4} \\ &= \frac{6x^4 + 4x^2 - 2}{(x^2+1)^4} = \frac{2(3x^2-1)(x^2+1)}{(x^2+1)^4} \end{aligned}$$

Let $f''(x) = 0$, then $3x^2 - 1 = 0$, so $x = \pm\sqrt{\frac{1}{3}} = \pm\frac{\sqrt{3}}{3}$, $f(\pm\sqrt{\frac{1}{3}}) = \frac{3}{4}$. So, points of inflection $(\frac{\sqrt{3}}{3}, \frac{3}{4})$ and $(-\frac{\sqrt{3}}{3}, \frac{3}{4})$.

(ii) f has no vertical asymptotes since $f(x)$ is defined on all $x \in \mathcal{R}$. For horizontal asymptotes, we check the following limits:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x^2+1} = 0,$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{x^2+1} = 0.$$

Therefore, the line $y = 0$ are horizontal asymptotes of the graph of f.

(iii)

x	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$
$f(x)$	$\frac{3}{4}$	1	$\frac{3}{4}$
$f'(x)$	$\nearrow$		$\searrow$
$f''(x)$	up	down	up