Calculus Midterm #2 (Form C)

(1)

$$f(x)=\sqrt{|2-x|}=\begin{cases}\sqrt{x-2}\,, & \text{if } x\geq 2;\\ \sqrt{2-x}\,, & \text{if } x<2.\end{cases}$$

(i) First, $f(2)=0$ is defined. For the limit at $x=2$, we check

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}\sqrt{x-2}=0,$$

and

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}\sqrt{2-x}=0,$$

which implies

$$\lim_{x\to 2} f(x)=0=f(2).$$

So, f is continuous at $x=2$.

(ii) By either

$$\lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2^+}\frac{\sqrt{x-2}-0}{x-2}=\lim_{x\to 2^+}\frac{1}{\sqrt{x-2}}=\infty$$

or

$$\lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2^-}\frac{\sqrt{2-x}-0}{x-2}=\lim_{x\to 2^-}\frac{-1}{\sqrt{2-x}}=-\infty.$$

We know f is *not* differentiable at $x=2$. ∎

(2) Yes. For a concave-up curve, it lies above its tangent line. Conversely, if the curve is concave down, it lies below its tangent line. Since a function changes the concavity at points of inflection, the tangent line definitely cross the graph of the function. ∎

(3) By implicit differentiation,

$$\frac{d}{dx}(y^2)=\frac{d}{dx}(\frac{x^3}{4-x}).$$

Hence,

$$2yy'=\frac{3x^2(4-x)-x^3\cdot(-1)}{(4-x)^2}.$$

Plug in $x=2$ and $y=2$ to obtain $4y'=\frac{12\cdot 2-8(-1)}{4}=8$, i.e., $y'=2$. Therefore, the slope of the curve is 2. ∎

(4)

(i) We have

$$\frac{dp}{dx}=\frac{1}{3}(9-x^3)^{-\frac{2}{3}}(-3x^2)=-x^2(9-x^3)^{-\frac{2}{3}}.$$

Then,

$$\eta=\frac{p/x}{dp/dx}=\frac{(9-x^3)^{\frac{1}{3}}x^{-1}}{-x^2(9-x^3)^{-\frac{2}{3}}}=-\frac{9-x^3}{x^3}=\frac{x^3-9}{x^3}.$$

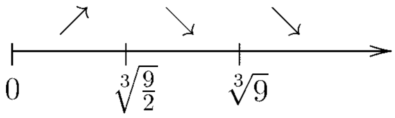
Let $x=2$, then $|\eta|=|\frac{8-9}{8}|=\frac{1}{8}<1$. Therefore, the demand is inelastic. For an economic interpretation, a 1% decrease in price results in an 0.125%

increase in the demand quantity at $x = 2$. That is a decrease in price is not accompanied by an increase in unit sales.

(ii) The total revenue $R = px = x(9 - x^3)^{\frac{1}{3}}$. Hence,

$$\begin{aligned} R' &= (9 - x^3)^{\frac{1}{3}} + x \cdot \frac{1}{3}(9 - x^3)^{-\frac{2}{3}}(-3x^2) \\ &= (9 - x^3)^{\frac{1}{3}} - x^3(9 - x^3)^{-\frac{2}{3}} \\ &= (9 - x^3)^{\frac{-2}{3}}[(9 - x^3) - x^3] = \frac{9 - 2x^3}{(9 - x^3)^{\frac{2}{3}}} \end{aligned}$$

Consider $x = \sqrt[3]{9}$ and $x = \sqrt[3]{\frac{9}{2}}$, for x-values in the interval $(0, \sqrt[3]{\frac{9}{2}})$, $R'(x) > 0$; for x-values in the interval $(\sqrt[3]{\frac{9}{2}}, \sqrt[3]{9})$, $R'(x) < 0$; for x-values in the interval $(\sqrt[3]{9}, \infty)$, $R'(x) < 0$. That is,



Therefore, $x^* = \sqrt[3]{\frac{9}{2}}$, we obtain a maximum total revenue. Then $p^* = \sqrt[3]{9 - x^{*3}} = \sqrt[3]{\frac{9}{2}}$. So, $(x^*, p^*) = (\sqrt[3]{\frac{9}{2}}, \sqrt[3]{\frac{9}{2}})$.

(iii) $x^* = 1.651$, then

$$|\eta| = |\frac{(\sqrt[3]{\frac{9}{2}})^3 - 9}{(\sqrt[3]{\frac{9}{2}})^3}| = | - 1| = 1.$$

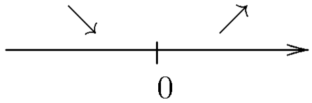
So, the demand at x^* is of unit elastic. Let $|\eta| < 1$, then $|\frac{x^3 - 9}{x^3}| < 1$. Since $\eta = \frac{p/x}{dp/dx} = \frac{x^3-9}{x^3} < 0$, we need to solve $-\frac{x^3-9}{x^3} < 1$, then $x^3 - 9 > -x^3$, which gives $x > \sqrt[3]{\frac{9}{2}} = x^*$. Therefore, for x-values in the interval $(x^*, 3)$, the demand is inelastic and by (ii) the total revenue is decreasing. ∎

(5)

(i)

$$f'(x) = \frac{2x}{(x^2 + 1)^2}.$$

We obtain the critical number $x = 0$. For $x \in (-\infty, 0)$, $f'(x) < 0$; for $x \in (0, \infty)$, $f'(x) > 0$. That is,



Therefore, when $x = 0$, we obtain relative minimum $f(0) = -1$.

$$\begin{aligned} f''(x) &= \frac{2(x^2 + 1)^2 - 2x \cdot 2(x^2 + 1) \cdot 2x}{(x^2 + 1)^4} \\ &= \frac{-6x^4 - 4x^2 + 2}{(x^2 + 1)^4} = \frac{-2(3x^2 - 1)}{(x^2 + 1)^3} \end{aligned}$$

Let $f''(x) = 0$, then $3x^2 - 1 = 0$, so $x = \pm\sqrt{\frac{1}{3}} = \pm\frac{\sqrt{3}}{3}$, $f(\pm\sqrt{\frac{1}{3}}) = \frac{-1}{\frac{1}{3}+1} = -\frac{3}{4}$. So, points of inflection $(\frac{\sqrt{3}}{3}, -\frac{3}{4})$ and $(-\frac{\sqrt{3}}{3}, -\frac{3}{4})$.

(ii) f has no vertical asymptotes since $f(x)$ is defined on all $x \in \mathcal{R}$. For horizontal asymptotes, we check the following limits:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{-1}{x^2+1} = 0,$$

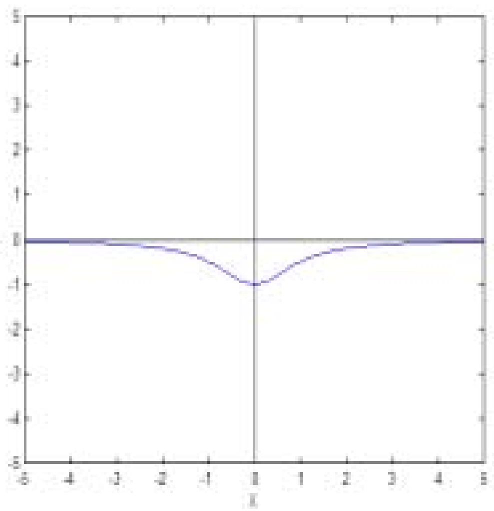
and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{-1}{x^2+1} = 0.$$

Therefore, the line $y = 0$ is a horizontal asymptote for the graph of f.

(iii)

x	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$
$f(x)$	$-\frac{3}{4}$	-1	$-\frac{3}{4}$
$f'(x)$	↘		↗
$f''(x)$	down	up	down



(6)

(i) We have

$$C = C(t) = \frac{3t}{27+t^3}.$$

Hence,

$$\Delta C = C(1.5) - C(1) = \frac{3\cdot 1.5}{27+1.5^3} - \frac{3\cdot 1}{27+1^3} \approx 0.041.$$

(ii)

$$\frac{dC}{dt} = \frac{3(27+t^3) - 3t(3t^2)}{(27+t^3)^2} = \frac{-6t^3+81}{(27+t^3)^2}.$$

Then,

$$dC = [\frac{-6t^3+81}{(27+t^3)^2}]dt.$$

Let $t = 1$ and $dt = 0.5$. Then,

$$dC = (\frac{-6^3+81}{(27+1^3)^2}) \cdot 0.5 \approx 0.048.$$

∎