

CALCULUS MIDTERM 1 SOLUTION

Exam Set: D

1.

(1) We have $f'(x) = (3x^2 + 2) + x \cdot 6x$. Hence, $f'(1) = 5 + 1 \cdot 6 = 11$.

(2) We have $f'(t) = -4t + 3$ and $f''(t) = -4$.

2.

(1) We have

$$\frac{d}{dx}(x^2y^3 - y^2 - xy - 1) = \frac{d}{dx}0.$$

Hence,

$$2xy^3 + 3x^2y^2y' - 2yy' - y - xy' = 0.$$

Let $x = -1$ and $y = 1$. Then $-2 + 3y' - 2y' - 1 + y' = 0$. Hence, $y' = \frac{3}{2}$. Then an equation of the tangent line is $y - 1 = \frac{3}{2}(x + 1)$ i.e., $y = \frac{3}{2}x + \frac{5}{2}$.

(2) We have $\frac{d}{dx}(x^{\frac{2}{3}} + y^{\frac{2}{3}}) = \frac{d}{dx}1$. Then $\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}y' = 0$. Hence, $y' = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$. Take the second derivative:

$$\begin{aligned} y'' &= -\frac{1}{3}\left(\frac{y}{x}\right)^{-\frac{2}{3}}\left(\frac{y'x - y}{x^2}\right) \\ &= -\frac{1}{3}y^{-\frac{2}{3}}x^{\frac{2}{3}} \cdot x^{-2}\left(-y^{\frac{1}{3}}x^{\frac{2}{3}} - y\right) \\ &= \frac{1}{3}x^{-\frac{2}{3}}y^{-\frac{1}{3}} + \frac{1}{3}x^{-\frac{4}{3}}y^{\frac{1}{3}} \end{aligned}$$

3.

(1) We have $\lim_{x \rightarrow 3^-} ax + b = -2$ and $\lim_{x \rightarrow -1^+} ax + b = 2$. Therefore, $a \cdot (-1) + b = 2$ and $a \cdot 3 + b = -2$. Hence $a = -1$ and $b = 1$.

(2)

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \frac{\sqrt{t+3+\Delta t} - \sqrt{t+3}}{\Delta t} &= \lim_{\Delta t \rightarrow 0} \frac{\sqrt{t+3+\Delta t} - \sqrt{t+3}}{\Delta t} \cdot \frac{\sqrt{t+3+\Delta t} + \sqrt{t+3}}{\sqrt{t+3+\Delta t} + \sqrt{t+3}} \\ &= \lim_{\Delta t \rightarrow 0} \frac{t+3+\Delta t - (t+3)}{\Delta t \cdot (\sqrt{t+3+\Delta t} + \sqrt{t+3})} \\ &= \frac{1}{\sqrt{t+3} + \sqrt{t+3}} \\ &= \frac{1}{2\sqrt{t+3}} \\ &= \frac{\sqrt{t+3}}{2(t+3)}, \quad t \neq -3 \end{aligned}$$

4.

(2) We have

$$h'(x) = \frac{(f(x) + xf'(x))(x + g(x)) - xf(x) \cdot (1 + g'(x))}{(x + g(x))^2}.$$

Hence,

$$h'(1) = \frac{(-2 + 1 \cdot (1)) \cdot (1 + (-2)) - 1 \cdot (-2) \cdot (1 + 3)}{(1 + (-2))^2} = 9.$$

5.

(1) False. The left-hand side is

$$\lim_{x \rightarrow 1} \left(\frac{2x}{x-1} - \frac{2}{x-1} \right) = \lim_{x \rightarrow 1} \frac{2x-2}{x-1} = 2.$$

But $\lim_{x \rightarrow -1} \frac{2}{x+1}$ does not exist.

(2) False. Consider the function $f(x) = |x|$. We have $\lim_{x \rightarrow 0} f(x) = 0 = f(0)$. Hence, f is continuous at $x = 0$. On the other hand,

$$\lim_{\Delta x \rightarrow 0} \frac{f(\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{|\Delta x|}{\Delta x}$$

which does not exist. Hence, $f(x)$ is not differentiable at $x = 0$.

6.

(1)

$$C' = 90 \cdot \frac{3}{2} \cdot \frac{1}{\sqrt{x}} = \frac{135}{\sqrt{x}} = \frac{135\sqrt{x}}{x} \quad x \neq 0.$$

(2)

$$\frac{dy}{dt} = 2x \frac{dx}{dt} = 2 \cdot 2 \cdot 2 = 8.$$

7.

(1)

$$\frac{dV}{dD} = L \cdot 2 \cdot \left(\frac{D-4}{4} \right) \cdot \frac{1}{4} \Big|_{L=12, D=16} = 18$$

(2)

$$\frac{dT}{dt} = \frac{-1300(2t+2)}{(t^2+2t+25)^2} \Big|_{t=3} = -\frac{13}{2} = -6.5.$$

8. We have $v = s' = 3t^2 - 12t - 10$ and $a = v' = 6t - 12$.

(1) Solve $3t^2 - 12t - 10 = 5$. We get $t = 5$ or -1 (negative number is not allowed). Hence, $t = 5$.

(2) We get $6t - 12 = 0$. Hence, $t = 2$.

9. We have

$$p(2) = 50 \cdot \frac{4+4+4}{4+4+8} = 30.$$

and

$$\frac{dp}{dt}(2) = 50 \cdot \frac{(2t+2)(t^2+4t+8) - (t^2+2t+4)(2t+4)}{(t^2+4t+8)^2} \Big|_{t=2} = 3.$$

Then

$$\begin{aligned} \frac{dR}{dt} &= 1000 \cdot \frac{p'(p+2) - (p+4)p'}{(p+2)^2} \Big|_{p=30, p'=3} \\ &= 1000 \cdot \frac{3 \cdot (30+2) - (30+4) \cdot 3}{(30+2)^2} \\ &= -\frac{375}{64} \\ &\approx -5.859 \end{aligned}$$

10. Let x be the distance from the bottom of the ladder to the wall and y be the height from the top of the ladder to the ground. When $x = 16$, then $16^2 + y^2 = 20^2$. Hence, $y = 12$. Now

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(20^2).$$

Then

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0.$$

Hence,

$$\frac{dy}{dt} = \frac{-2x \frac{dx}{dt}}{2y} = -\frac{20}{3} \approx -6.67.$$