CALCULUS MIDTERM 1 SOLUTION

Exam Set: C

1.

(1) We have $\lim_{x\to 3^-} cx+d = -4$ and $\lim_{x\to -1^+} cx+d = 4$. Therefore, $c\cdot(-1)+d = 4$ and $c\cdot 3+d = -4$. Hence $c = -2$ and $d = 2$.

(2)

$$\begin{aligned}
\lim_{\triangle x\to 0} \frac{\sqrt{x-1+\triangle x}-\sqrt{x-1}}{\triangle x} &= \lim_{\triangle x\to 0} \frac{\sqrt{x-1+\triangle x}-\sqrt{x-1}}{\triangle x}\cdot\frac{\sqrt{x-1+\triangle x}+\sqrt{x-1}}{\sqrt{x-1+\triangle x}+\sqrt{x-1}} \\
&= \lim_{\triangle x\to 0} \frac{x-1+\triangle x-(x-1)}{\triangle x\cdot(\sqrt{x-1+\triangle x}+\sqrt{x-1})} \\
&= \frac{1}{\sqrt{x-1}+\sqrt{x-1}} \\
&= \frac{1}{2\sqrt{x-1}} \\
&= \frac{\sqrt{x-1}}{2(x-1)}, \quad x \neq 1
\end{aligned}$$

2.

(1) $f'(x) = 1(x^2+1)+(x+1)(2x)$. Hence, $f'(1) = 2+2\cdot 2 = 6$.

(2) $f'(t) = -2t+4$ and $f''(t) = -2$.

3.

(1) We have

$$\frac{d}{dx}(x^2y^3 - y^2 + xy - 1) = \frac{d}{dx}0$$

Hence,

$$2xy^3 + 3x^2y^2y' - 2yy' + y + xy' = 0.$$

Let $x = 1$ and $y = 1$. Then $2+3y'-2y'+1+y' = 0$. Hence, $y' = -\frac{3}{2}$. Then an equation of the tangent line is $y-1 = -\frac{3}{2}(x-1)$, i.e., $y = -\frac{3}{2}x+\frac{5}{2}$.

(2) We have $\frac{d}{dx}(x^{\frac{1}{3}}+y^{\frac{1}{3}}) = \frac{d}{dx}(1)$. Then $\frac{1}{3}x^{-\frac{2}{3}}+\frac{1}{3}y^{-\frac{2}{3}}y' = 0$. Hence, $y' = -\left(\frac{y}{x}\right)^{\frac{2}{3}}$. Take the second derivative:

$$\begin{aligned}
y'' &= -\frac{2}{3}\left(\frac{y}{x}\right)^{-\frac{1}{3}}\left(\frac{y'x-y}{x^2}\right) \\
&= -\frac{2}{3}y^{-\frac{1}{3}}x^{\frac{1}{3}}\cdot x^{-2}\left(-y^{\frac{2}{3}}x^{\frac{1}{3}}-y\right) \\
&= \frac{2}{3}x^{-\frac{4}{3}}y^{\frac{1}{3}}+\frac{2}{3}x^{-\frac{5}{3}}y^{\frac{2}{3}}.
\end{aligned}$$

4.

(2) We have

$$h'(x) = \frac{(f(x)+xf'(x))(x-g(x)) - xf(x)\cdot(1-g'(x))}{(x-g(x))^2}$$

Hence,

$$h'(1) = \frac{(2+1\cdot(-1))\cdot(1-(-2)) - 1\cdot 2\cdot(1-3)}{(1-(-1))^2} = \frac{7}{9}$$

5.

(1) False. The left-hand side is

$$\lim_{x\to 1}\left(\frac{2x}{x-1}-\frac{2}{x-1}\right)=\lim_{x\to 1}\frac{2x-2}{x-1}=2.$$

But $\lim_{x\to -1}\frac{2}{x+1}$ does not exist.

(2) True. If f is differentiable at $x=a$, then $\lim_{\Delta x\to 0}\frac{f(a+\Delta x)-f(a)}{\Delta x}$ exists. Therefore $\lim_{\Delta x\to 0} f(a+\Delta x)=f(a)$. Hence, f is continuous at $x=a$.

6.

(1)

$$C' = 100\cdot 2\cdot \frac{1}{2}\frac{1}{\sqrt{x}}=\frac{100}{\sqrt{x}}=\frac{100\sqrt{x}}{x},\quad x\neq 0.$$

(2)

$$\frac{dy}{dt}=2x\frac{dx}{dt}=2\cdot 3\cdot 2=12.$$

7. We have $s=t^3-4.5t^2-7t$. Then $v=s'=3t^2-9t-7$ and $a=v'=6t-9$.

(1) Solve $3t^2-9t-7=5$. We have $3t^2-9t-12=0$. Hence, $t=4$.

(2) Solve $6t-9=0$. We get $t=\frac{3}{2}=1.5$.

8.

(1)

$$\frac{dV}{dD}=L\cdot 2\cdot\left(\frac{D-4}{4}\right)\cdot\frac{1}{4}\bigg|_{L=12,D=8}=6.$$

(2)

$$\frac{dT}{dt}=\frac{-1300(2t+2)}{(t^2+2t+25)^2}\bigg|_{t=5}=-\frac{13}{3}\approx -4.33.$$

9. We have

$$p(16)=\frac{400}{1+\frac{1}{8}\sqrt{16}}+200=\frac{1400}{3}.$$

Then

$$\frac{dp}{dt}=\frac{-400\cdot\frac{1}{8}\cdot\frac{1}{2\sqrt{t}}}{\left(1+\frac{1}{8\sqrt{t}}\right)^2}\Bigg|_{t=16}=-\frac{25}{9}.$$

Then

$$\frac{dx}{dt}=\frac{100}{9}\cdot\frac{1}{2}\frac{-2pp'}{\sqrt{810000-p^2}}\Bigg|_{p=\frac{1400}{3},\ p'=-\frac{25}{9}}\approx 18.7163.$$

10. Let x be the distance from the bottom of the ladder to the wall and y be the height from the top of the ladder to the ground. When $x=16$, then $12^2+y^2=20^2$. Hence, $y=16$. Now

$$\frac{d}{dt}(x^2+y^2)=\frac{d}{dt}(20^2).$$

Then

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0.$$

Hence,

$$\frac{dy}{dt}=\frac{-2x\frac{dx}{dt}}{2y}=-\frac{15}{4}\approx -3.75.$$