CALCULUS MIDTERM 1 SOLUTION

Exam Set: B

1.

(1) We have $f'(x) = (3x^2+2) + x\cdot 6x$. Hence, $f'(1) = 5 + 1\cdot 6 = 11$.

(2) We have $f'(t) = -4t+3$ and $f''(t) = -4$.

2.

(1) We have $\lim_{x\to 3^-} ax+b = -2$ and $\lim_{x\to -1^+} ax+b = 2$. Therefore, $a\cdot(-1)+b = 2$ and $a\cdot 3+b=-2$. Hence $a=-1$ and $b=1$.

(2)

$$\begin{aligned}
\lim_{\triangle t\to 0}\frac{\sqrt{t+3+\triangle t}-\sqrt{t+3}}{\triangle t} &= \lim_{\triangle t\to 0}\frac{\sqrt{t+3+\triangle t}-\sqrt{t+3}}{\triangle t}\cdot\frac{\sqrt{t+3+\triangle t}+\sqrt{t+3}}{\sqrt{t+3+\triangle t}+\sqrt{t+3}}\\
&= \lim_{\triangle t\to 0}\frac{t+3+\triangle t-(t+3)}{\triangle t\cdot(\sqrt{t+3+\triangle t}+\sqrt{t+3})}\\
&= \frac{1}{\sqrt{t+3}+\sqrt{t+3}}\\
&= \frac{1}{2\sqrt{t+3}}\\
&= \frac{\sqrt{t+3}}{2(t+3)},\quad t\neq -3
\end{aligned}$$

3.

(2) We have

$$h'(x) = \frac{(f(x)+xf'(x))(x+g(x)) - xf(x)\cdot(1+g'(x))}{(x+g(x))^2}.$$

Hence,

$$h'(1) = \frac{(-2+1\cdot(1))\cdot(1+(-2)) - 1\cdot(-2)\cdot(1+3)}{(1+(-2))^2} = 9.$$

4.

(1) We have

$$\frac{d}{dx}(x^2y^3 - y^2 - xy - 1) = \frac{d}{dx}0.$$

Hence,

$$2xy^3 + 3x^2y^2y' - 2yy' - y - xy' = 0.$$

Let $x=-1$ and $y=1$. Then $-2+3y'-2y'-1+y'=0$. Hence, $y'=\frac{3}{2}$. Then an equation of the tangent line is $y-1=\frac{3}{2}(x+1)$ i.e., $y=\frac{3}{2}x+\frac{5}{2}$.

(2) We have $\frac{d}{dx}(x^{\frac{2}{3}}+y^{\frac{2}{3}}) = \frac{d}{dx}1$. Then $\frac{2}{3}x^{-\frac{1}{3}}+\frac{2}{3}y^{-\frac{1}{3}}y'=0$. Hence, $y' = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$. Take the second derivative:

$$\begin{aligned}
y'' &= -\tfrac{1}{3}\left(\tfrac{y}{x}\right)^{-\frac{2}{3}}\left(\tfrac{y'x-y}{x^2}\right)\\
&= -\tfrac{1}{3}y^{-\frac{2}{3}}x^{\frac{2}{3}}\cdot x^{-2}\left(-y^{\frac{1}{3}}x^{\frac{2}{3}}-y\right)\\
&= \tfrac{1}{3}x^{-\frac{2}{3}}y^{-\frac{1}{3}}+\tfrac{1}{3}x^{-\frac{4}{3}}y^{\frac{1}{3}}
\end{aligned}$$

5.

(1)

$$C' = 90\cdot\frac{3}{2}\cdot\frac{1}{\sqrt{x}} = \frac{135}{\sqrt{x}} = \frac{135\sqrt{x}}{x}\ x\neq 0.$$

(2)

$$\frac{dy}{dt} = 2x\frac{dx}{dt} = 2 \cdot 2 \cdot 2 = 8.$$

6.

(1) False. The left-hand side is

$$\lim_{x\to 1}\left(\frac{2x}{x-1} - \frac{2}{x-1}\right) = \lim_{x\to 1}\frac{2x-2}{x-1} = 2.$$

But $\lim_{x\to -1}\frac{2}{x-1}$ does not exist.

(2) True. If f is differentiable at $x = a$, then $\lim_{\Delta x\to 0}\frac{f(a+\Delta x)-f(a)}{\Delta x}$ exists. Therefore $\lim_{\Delta x\to 0} f(a+\Delta x) = f(a)$. Hence, f is continuous at $x = a$.

7. We have $s = t^3 - 4.5t^2 - 7t$. Then $v = s' = 3t^2 - 9t - 7$ and $a = v' = 6t - 9$.

(1) Solve $3t^2 - 9t - 7 = 5$. We have $3t^2 - 9t - 12 = 0$. Hence, $t = 4$.

(2) Solve $6t - 9 = 0$. We get $t = \frac{3}{2} = 1.5$.

8. We have

$$p(16) = \frac{400}{1+\frac{1}{8}\sqrt{16}} + 200 = \frac{1400}{3}.$$

Then

$$\frac{dp}{dt} = \left.\frac{-400\cdot\frac{1}{8}\cdot\frac{1}{2\sqrt{t}}}{\left(1+\frac{1}{8\sqrt{t}}\right)^2}\right|_{t=16} = -\frac{25}{9}.$$

Then

$$\frac{dx}{dt} = \frac{100}{9}\cdot\frac{1}{2}\left.\frac{-2pp'}{\sqrt{810000-p^2}}\right|_{p=\frac{1400}{3},\ p'=-\frac{25}{9}} \approx 18.7163.$$

9.

(1)

$$\frac{dV}{dD} = L\cdot 2\cdot\left(\frac{D-4}{4}\right)\cdot\left.\frac{1}{4}\right|_{L=12, D=8} = 6.$$

(2)

$$\frac{dT}{dt} = \left.\frac{-1300(2t+2)}{(t^2+2t+25)^2}\right|_{t=5} = -\frac{13}{3} \approx -4.33.$$

10. Let x be the distance from the bottom of the ladder to the wall and y be the height from the top of the ladder to the ground. When $x = 16$, then $12^2 + y^2 = 20^2$. Hence, $y = 16$. Now

$$\frac{d}{dt}(x^2+y^2) = \frac{d}{dt}(20^2).$$

Then

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

Hence,

$$\frac{dy}{dt} = \frac{-2x\frac{dx}{dt}}{2y} = -\frac{15}{4} \approx -3.75.$$