CALCULUS MIDTERM 1 SOLUTION

Exam Set: A

1.

(1) $f'(x) = 1(x^2+1) + (x+1)(2x)$. Hence, $f'(1) = 2 + 2 \cdot 2 = 6$.

(2) $f'(t) = -2t + 4$ and $f''(t) = -2$.

2.

(1) We have $\lim_{x \to 3^-} cx + d = -4$ and $\lim_{x \to -1^+} cx + d = 4$. Therefore, $c \cdot (-1) + d = 4$ and $c \cdot 3 + d = -4$. Hence $c = -2$ and $d = 2$.

(2)

$$\begin{aligned}
\lim_{\triangle x \to 0} \frac{\sqrt{x-1+\triangle x} - \sqrt{x-1}}{\triangle x} &= \lim_{\triangle x \to 0} \frac{\sqrt{x-1+\triangle x} - \sqrt{x-1}}{\triangle x} \cdot \frac{\sqrt{x-1+\triangle x} + \sqrt{x-1}}{\sqrt{x-1+\triangle x} + \sqrt{x-1}} \\
&= \lim_{\triangle x \to 0} \frac{x - 1 + \triangle x - (x-1)}{\triangle x \cdot (\sqrt{x-1+\triangle x} + \sqrt{x-1})} \\
&= \frac{1}{\sqrt{x-1} + \sqrt{x-1}} \\
&= \frac{1}{2\sqrt{x-1}} \\
&= \frac{\sqrt{x-1}}{2(x-1)}, \quad x \neq 1
\end{aligned}$$

3.

(2) We have

$$h'(x) = \frac{(f(x) + xf'(x))(x - g(x)) - xf(x) \cdot (1 - g'(x))}{(x - g(x))^2}$$

Hence,

$$h'(1) = \frac{(2 + 1 \cdot (-1)) \cdot (1 - (-2)) - 1 \cdot 2 \cdot (1 - 3)}{(1 - (-1))^2} = \frac{7}{9}$$

4.

(1) We have

$$\frac{d}{dx}(x^2y^3 - y^2 + xy - 1) = \frac{d}{dx}0$$

Hence,

$$2xy^3 + 3x^2y^2y' - 2yy' + y + xy' = 0.$$

Let $x = 1$ and $y = 1$. Then $2 + 3y' - 2y' + 1 + y' = 0$. Hence, $y' = -\frac{3}{2}$. Then an equation of the tangent line is $y - 1 = -\frac{3}{2}(x-1)$ i.e., $y = -\frac{3}{2}x + \frac{5}{2}$.

(2) We have $\frac{d}{dx}(x^{\frac{1}{3}} + y^{\frac{1}{3}}) = \frac{d}{dx}(1)$. Then $\frac{1}{3}x^{-\frac{2}{3}} + \frac{1}{3}y^{-\frac{2}{3}}y' = 0$. Hence, $y' = -\left(\frac{y}{x}\right)^{\frac{2}{3}}$. Take the second derivative:

$$\begin{aligned}
y'' &= -\frac{2}{3}\left(\frac{y}{x}\right)^{-\frac{1}{3}}\left(\frac{y'x - y}{x^2}\right) \\
&= -\frac{2}{3}y^{-\frac{1}{3}}x^{\frac{1}{3}} \cdot x^{-2}\left(-y^{\frac{2}{3}}x^{\frac{1}{3}} - y\right) \\
&= \frac{2}{3}x^{-\frac{4}{3}}y^{\frac{1}{3}} + \frac{2}{3}x^{-\frac{5}{3}}y^{\frac{2}{3}}.
\end{aligned}$$

5.

(1)

$$C' = 100 \cdot 2 \cdot \frac{1}{2}\frac{1}{\sqrt{x}} = \frac{100}{\sqrt{x}} = \frac{100\sqrt{x}}{x}, \quad x \neq 0.$$

(2)

$$\frac{dy}{dt} = 2x\frac{dx}{dt} = 2 \cdot 3 \cdot 2 = 12.$$

6.

(1) False. The left-hand side is

$$\lim_{x\to 1}\left(\frac{2x}{x+1} + \frac{2}{x+1}\right) = \lim_{x\to 1}\frac{2x+2}{x+1} = 2.$$

But $\lim_{x\to -1}\frac{2}{x+1}$ does not exist.

(2) False. Consider the function $f(x) = |x|$. We have $\lim_{x\to 0} f(x) = 0 = f(0)$. Hence, f is continuous at $x = 0$. On the other hand,

$$\lim_{\Delta x\to 0}\frac{f(\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x\to 0}\frac{|\Delta x|}{\Delta x}$$

which does not exist. Hence, $f(x)$ is not differentiable at $x = 0$.

7. We have $v = s' = 3t^2 - 12t - 10$ and $a = v' = 6t - 12$.

(1) Solve $3t^2 - 12t - 10 = 5$. We get $t = 5$ or -1 (negative number is not allowed). Hence, $t = 5$.

(2) We get $6t - 12 = 0$. Hence, $t = 2$.

8. We have

$$p(2) = 50 \cdot \frac{4+4+4}{4+4+8} = 30.$$

and

$$\frac{dp}{dt}(2) = 50 \cdot \left.\frac{(2t+2)(t^2+4t+8) - (t^2+2t+4)(2t+4)}{(t^2+4t+8)^2}\right|_{t=2} = 3.$$

Then

$$\begin{aligned}\tfrac{dR}{dt} &= 1000 \cdot \left.\tfrac{p'(p+2)-(p+4)p'}{(p+2)^2}\right|_{p=30,p'=3} \\ &= 1000 \cdot \tfrac{3\cdot(30+2)-(30+4)\cdot 3}{(30+2)^2} \\ &= -\tfrac{375}{64} \\ &\approx -5.859\end{aligned}$$

9.

(1)

$$\frac{dV}{dD} = L \cdot 2 \cdot \left(\frac{D-4}{4}\right) \cdot \left.\frac{1}{4}\right|_{L=12,D=16}. = 18$$

(2)

$$\frac{dT}{dt} = \left.\frac{-1300(2t+2)}{(t^2+2t+25)^2}\right|_{t=3} = -\frac{13}{2} = -6.5.$$

10. Let x be the distance from the bottom of the ladder to the wall and y be the height from the top of the ladder to the ground. When $x = 16$, then $16^2 + y^2 = 20^2$. Hence, $y = 12$. Now

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(20^2).$$

Then

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

Hence,

$$\frac{dy}{dt} = \frac{-2x\frac{dx}{dt}}{2y} = -\frac{20}{3} \approx -6.67.$$