# CALCULUS Final SOLUTION

Exam Set: D

1. (i) Let $u = 1 + cos^2 t$, $du = -2sintcostdt$,Then the integral

$$\int \frac{sintcost}{\sqrt{1+cos^2 t}}dt = -\frac{1}{2}\int \frac{1}{\sqrt{u}}du = -u^{1/2} + C = -\sqrt{1+cos^2 t} + C$$

(ii)

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^x}{(1+e^x)^3}dx &= \lim_{a\to-\infty}\int_a^0 \frac{e^x}{(1+e^x)^3}dx + \lim_{b\to\infty}\int_0^b \frac{e^x}{(1+e^x)^3}dx \\
(u = 1+e^x, x: a \to 0, x: 0 \to b &\Rightarrow du = e^x dx, u: 1+e^a \to 2, u: 2 \to 1+e^b) \\
&= \lim_{a\to-\infty}\int_{1+e^a}^{2} \frac{1}{u^3}du + \lim_{b\to\infty}\int_2^{1+e^b} \frac{1}{u^3}du \\
&= \lim_{a\to-\infty}\left[-\frac{1}{2u^2}\right]_{1+e^a}^{2} + \lim_{b\to\infty}\left[-\frac{1}{2u^2}\right]_2^{1+e^b} \\
&= \lim_{a\to-\infty}\left[-\frac{1}{8} + \frac{1}{2(1+e^a)^2}\right] + \lim_{b\to\infty}\left[-\frac{1}{2(1+e^b)^2} + \frac{1}{8}\right] \\
&= -\frac{1}{8} + \lim_{a\to-\infty}\frac{1}{2(1+e^a)^2} - \lim_{b\to\infty}\frac{1}{2(1+e^b)^2} + \frac{1}{8} \\
&= -\frac{1}{8} + \frac{1}{2} + \frac{1}{8} = \frac{1}{2}
\end{aligned}$$

2. (i) To find the first partial derivative with respect to $y$,hold $x$ constant to obtain

$$f_y(x,y) = \frac{\partial}{\partial y}\sqrt{x^2+y^2} = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}$$

The value of $f_y(x,y)$ at the point $(6,8)$ is

$$f_y(6,8) = \frac{8}{\sqrt{(6)^2+(8)^2}} = \frac{8}{\sqrt{100}} = \frac{8}{10} = \frac{4}{5} = 0.8$$

(ii) Since $\lim_{x\to\infty}\frac{\ln x}{x} = \frac{\infty}{\infty}$, you can apply L'Hopital's Rule,as follows

$$\lim_{x\to\infty}\frac{lnx}{x} = \lim_{x\to\infty}\frac{\frac{d}{dx}[lnx]}{\frac{d}{dx}[x]} = \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = \lim_{x\to\infty}\frac{1}{x} = 0$$

3. (i) First you interchange the order of integration so that $y$ is the outer variable ,then $y$ will have constant bounds of integration given by $0 \leq y \leq 1$.Solving for $x$ in the equation $y = \sqrt{x}$ implies that the bounds for $x$ are $0 \leq x \leq y^2$.Thus

$$\begin{aligned}
\int_0^1 \int_{\sqrt{x}}^1 sin\left(\frac{y^3+1}{2}\right) dydx &= \int_0^1 \int_0^{y^2} sin\left(\frac{y^3+1}{2}\right) dxdy = \int_0^1 sin\left(\frac{y^3+1}{2}\right) y^2 dy \\
(u = \frac{y^3+1}{2}, y : 0 \to 1 &\Rightarrow du = \frac{3}{2}y^2 dy, u : 1/2 \to 1) \\
&= \int_{1/2}^1 sin(u)\frac{2}{3}du = \frac{2}{3}\int_{1/2}^1 sin(u)du = \frac{-2}{3}[cosu]_{1/2}^1 \\
&= \frac{-2}{3}[cos(1) - cos(1/2)]
\end{aligned}$$

(ii) By integration by parts, we can written the integral as follows

$$\int_0^3 xf''(x)dx = [xf'(x)]_0^3 - \int_0^3 f'(x)dx = [xf'(x)]_0^3 - [f(x)]_0^3 = 3f'(3) - f(3) + f(0)$$

By assumption $f(0) = 5, f(3) = 5$, and $f'(3) = 4$.Thus

$$\int_0^3 xf''(x)dx = 3(4) - 5 + 5 = 12$$

4. (i) $y' = y(1-y) \Rightarrow \frac{dy}{dt} = y(1-y) \Rightarrow \frac{1}{y(1-y)}dy = dt$

Integrate both sides $\int dt = \int \frac{1}{y(1-y)}dy$.Then

$$\begin{aligned}
t + C_1 &= \int \frac{1}{y}dy + \int \frac{1}{1-y}dy = ln \mid y \mid -ln \mid 1-y \mid = ln \mid \frac{y}{1-y} \mid \\
\Rightarrow \quad lnC_2e^t &= ln \mid \frac{y}{1-y} \mid \Rightarrow Ce^t = \frac{y}{1-y} \Rightarrow Ce^t - yCe^t = y \\
\Rightarrow \quad (1+Ce^t)y &= Ce^t \Rightarrow y(t) = \frac{Ce^t}{1+Ce^t}
\end{aligned}$$

(ii) $\frac{1}{3} = y(0) = \frac{Ce^{(0)}}{1+Ce^{(0)}} = \frac{C}{1+C} \Rightarrow C = \frac{1}{2}$.Thus the solution for the equation with initial condition is $y(t) = \frac{e^t}{2+e^t}$

5. (i)

$$\begin{aligned}
E[X] &= 0P(X=0) + 1P(X=1) + 2P(X=2) + 3P(X=3) + \cdots \\
&= e^{-2}\frac{2^1}{1!} + 2e^{-2}\frac{2^2}{2!} + 3e^{-2}\frac{2^3}{3!} + \cdots = 2e^{-2}(1 + e^{-2}\frac{2^1}{1!} + e^{-2}\frac{2^2}{2!} + \cdots) \\
&= 2e^{-2}\sum_{n=0}^{\infty}\frac{2^n}{n!} = 2e^{-2}e^2 = 2
\end{aligned}$$

(ii)

$$\begin{aligned}
\mu &= \int_0^\infty x\left(5e^{-5x}\right)dx = \lim_{a\to\infty}\int_0^a x\left(5e^{-5x}\right)dx \\
&= \lim_{a\to\infty}\left[-xe^{-5x}\big|_0^a + \int_0^a e^{-5x}dx\right] = \lim_{a\to\infty}\left[-ae^{-5a} - \frac{1}{5}e^{-5x}\Big|_0^a\right] \\
&= \lim_{a\to\infty}\left[-ae^{-5a} - \frac{1}{5e^{5a}} + \frac{1}{5}\right] = \lim_{a\to\infty}\frac{-a}{e^{-5a}} - \lim_{a\to\infty}\frac{1}{5e^{5a}} + \frac{1}{5} \\
&= -\lim_{a\to\infty}\frac{1}{5e^{5a}} + \frac{1}{5} = \frac{1}{5} \\
\int_0^\infty x^2\left(5e^{-5x}\right)dx &= \lim_{a\to\infty}\int_0^a x^2\left(5e^{-5x}\right)dx \\
&= \lim_{a\to\infty}\left[-x^2e^{-5x}\big|_0^a - \frac{2}{5}\,xe^{-5x}\big|_0^a + \frac{2}{5}\int_0^a e^{-5x}dx\right] \\
&= \lim_{a\to\infty}\left[\frac{-a^2}{e^{-5a}} - \frac{2a}{5e^{5a}} - \frac{2}{5}\,e^{-5x}\big|_0^a\right] \\
&= \lim_{a\to\infty}\frac{-a^2}{e^{5a}} - \lim_{a\to\infty}\frac{2a}{5e^{5a}} - \lim_{a\to\infty}\frac{2}{25e^{5a}} + \frac{2}{25} \\
&= -\lim_{a\to\infty}\frac{2a}{5e^{4a}} - \lim_{a\to\infty}\frac{2}{25e^{5a}} + \frac{2}{25} = -\lim_{a\to\infty}\frac{2}{25e^{5a}} + \frac{2}{25} \\
&= \frac{2}{25}
\end{aligned}$$

Thus

$$\sigma = \sqrt{\frac{2}{25} - (\frac{1}{5})^2} = \sqrt{\frac{1}{25}} = \frac{1}{5}$$

6. (i) Since $f(x) = ax^2(2-x)$ is a probability density function on the interval $[0, 2]$. So, evaluate the following integral

$$1 = \int_0^2 ax^2(2-x)dx = a\int_0^2 x^2(2-x)dx = a\left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2 = a\left[\frac{16}{3} - 4\right] = \frac{4a}{3}$$

Thus, $a = \frac{3}{4}$.

(ii)

$$\begin{aligned}
P\left(0 \le x \le \frac{1}{2}\right) &= \int_0^{1/2}\frac{3}{4}x^2(2-x)dx = \frac{3}{4}\int_0^{1/2}x^2(2-x)dx = \frac{3}{4}\left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^{1/2} \\
&= \frac{3}{4}\left[\frac{1}{12} - \frac{1}{64}\right] = \frac{13}{256}
\end{aligned}$$

7. (i) When the ball hits the ground the first time, it has traveled a distance of

$$D_1 = 18$$

.Between the first and second times it hits the ground, it has traveled an additional distance of

$$D_2 = 18\left(\frac{7}{10}\right) + 18\left(\frac{7}{10}\right) = 36\left(\frac{7}{10}\right)$$

Between the second and third times the ball hits the ground, it has traveled an additional distance of

$$D_3 = 18\left(\frac{7}{10}\right)\left(\frac{7}{10}\right) + 18\left(\frac{7}{10}\right)\left(\frac{7}{10}\right) = 36\left(\frac{7}{10}\right)^2$$

Continuing this process,you obtain a total vertical distance traveled of

$$\begin{aligned} D &= 18 + 36\left(\frac{7}{10}\right) + 36\left(\frac{7}{10}\right)^2 + \cdots = -18 + 36 + 36\left(\frac{7}{10}\right) + 36\left(\frac{7}{10}\right)^2 + \cdots \\ &= -18 + \sum_{n=0}^{\infty} 36\left(\frac{7}{10}\right)^n = -18 + \frac{36}{1 - \left(\frac{7}{10}\right)} = -18 + 120 = 102 \end{aligned}$$

(ii) The infinite series

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k^2}} = \frac{1}{1^{2/3}} + \frac{1}{2^{2/3}} + \frac{1}{3^{2/3}} + \cdots$$

is a $p$-series with $p = \frac{2}{3}$.Because $p < 1$,you can conclude that the series divergence.

8. (i) The first derivative of $f$ is $f'(x) = 3x^2$. Thus ,the iterative formula for Newton's Method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x^3 - 66}{3x^2}$$

The calculations for two iterations are shown in the table.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 4 | −2 | 48 | 0.04167 | 4.04167 |
| 2 | 4.04167 | 0.02107 | 49.00529 | 0.00043 | 4.0412 |
| 3 | 4.0412 | | | | |

Thus,the approximation is $\sqrt[3]{67} = 4.0412$

(ii) Begin by finding several derivatives of f and evaluation each at $c = 64$

$$\begin{aligned} g(x) &= x^{1/3} & g(64) &= 4 \\ g'(x) &= \frac{1}{3}x^{-2/3} & g'(64) &= \frac{1}{3}(0.0625) = 0.02083 \\ g''(x) &= \frac{-2}{9}x^{-5/3} & g''(64) &= \frac{-2}{9}(0.00098) = -0.00011 \end{aligned}$$

Thus, the two-degree Taylor polynomial is

$$g(x) = g(64) + g'(64)(x-64) + \frac{g''(64)(x-64)^2}{2!} = 4 + 0.02083(x-64) - 0.00011(x-64)^2$$

To evaluate the series when $x = 66$.

$$g(66) = 4+0.02083(66-64)-0.00011(66-64)^2 = 4+0.02083(2)-0.00011(4) = 4.04122$$

9. Let $T(x,y,z) = 8x^2yz$ and $g(x,y,z) = x^2+y^2+z^2-1$.Then, define a new function $F(x,y,x,\lambda)$ by

$$F(x,y,x,\lambda) = T(x,y,z) - \lambda(x,y,z) = 8x^2yz - \lambda\left(x^2+y^2+z^2-1\right)$$

To find the critical numbers of $F$, set the partial derivatives of $F$ with respect to $x$,$y$,$z$,and $\lambda$ equal to zero and obtain

$$F_x(x,y,x,\lambda) = 16xyz - 2\lambda x = 0, \qquad F_y(x,y,x,\lambda) = 8x^2z - 2\lambda y = 0,$$
$$F_z(x,y,x,\lambda) = 8x^2y - 2\lambda z = 0 \qquad , F_\lambda(x,y,x,\lambda) = -x^2-y^2-z^2+1 = 0$$

Then

$$\frac{F_x}{F_y} : \frac{2y}{x} = \frac{x}{y}, \frac{F_y}{F_z} : \frac{z}{y} = \frac{y}{z}, \frac{F_x}{F_z} : \frac{2z}{x} = \frac{x}{z}$$
$$\Rightarrow \quad x^2 = 2y^2, z^2 = y^2$$

Substitute this into the equation $F_\lambda(x,y,x,\lambda) = -x^2-y^2-z^2+1 = 0$ and solve $y$

$$0 = F_\lambda(x,y,x,\lambda) = -2y^2 - y^2 - y^2 + 1 \Rightarrow y^2 = \frac{1}{4} = \pm\frac{1}{2}$$

Using this $y$-value,you can conclude that the critical values are

$$y = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}, z = \pm\frac{1}{2}$$

$$x = \frac{-1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}, z = \pm\frac{1}{2}$$

which implies that the temperature value is

$$T\left(\pm\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}\right) = 8(1/2)(1/2)(1/2) = 1, T\left(\pm\frac{1}{\sqrt{2}}, -\frac{1}{2}, \frac{1}{2}\right) = 8(1/2)(-1/2)(1/2) = -1,$$
$$T\left(\pm\frac{1}{\sqrt{2}}, \frac{1}{2}, -\frac{1}{2}\right) = 8(1/2)(1/2)(-1/2) = -1, T\left(\pm\frac{1}{\sqrt{2}}, -\frac{1}{2}, -\frac{1}{2}\right) = 8(1/2)(1/2)(1/2) = 1,$$

Thus,the point(s) on the sphere at which the temperature is greatest is

$$\left(\pm\frac{1}{\sqrt{2}},\frac{1}{2},\frac{1}{2}\right),\left(\pm\frac{1}{\sqrt{2}},-\frac{1}{2},-\frac{1}{2}\right)$$

the point(s) on the sphere at which the temperature is least is

$$\left(\pm\frac{1}{\sqrt{2}},-\frac{1}{2},\frac{1}{2}\right),\left(\pm\frac{1}{\sqrt{2}},\frac{1}{2},-\frac{1}{2}\right)$$

10. Finding several derivatives of $f$ and evaluation each at $c = 0$

$$\begin{aligned}
f(x) &= ln\left(\tfrac{1+x}{1-x}\right) = ln(1+x) - ln(1-x) & f(0) &= 0\\
f'(x) &= \frac{1}{1+x} - \frac{-1}{1-x} = \frac{2}{1-x^2} & f'(0) &= 2\\
f''(x) &= \frac{0-(-2x)}{\left(1-x^2\right)^2} = \frac{2x}{\left(1-x^2\right)^2} & f''(0) &= 0\\
f^{(3)}(x) &= \frac{2\left(1-x^2\right)^2 - 2x(2(1-x^2)(-2x))}{\left(1-x^2\right)^4} = \frac{2+6x^2}{\left(1-x^2\right)^3} & f^{(3)}(0) &= 2
\end{aligned}$$

Continuing this process,we can see that $f^{(2n-2)}(0) = 0, f^{(2n-1)}(0) = 2$, Thus the Taylor series

$$\begin{aligned}
f(x) &= f(0) + f'(0)x + \frac{f'(0)x^2}{2!} + \frac{f'(0)x^3}{3!} + \frac{f'(0)x^4}{4!} + \cdots\\
&= 2x + \frac{2}{3!}x^3 + \frac{2}{5!}x^5 + \cdots = \sum_{n=0}^{\infty}\frac{2x^{2n+1}}{(2n+1)!}
\end{aligned}$$

For this power series, $a_n = \dfrac{2}{(2n+1)!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{2(n+1)+1}}{a_n x^{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{2}{(2n+3)!}x^{2n+3}}{\frac{2}{(2n+1)!}x^{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{x^2}{(2n+3)(2n+2)}\right| = 0$$

So,by the Ration Test, this series converges for all $x$ and the radius of convergence is $\infty$