CALCULUS Final SOLUTION

Exam Set: B

1. (i) To find the first partial derivative with respect to y,hold x constant to obtain

$$f_y(x,y) = \frac{\partial}{\partial y}\sqrt{x^2+y^2} = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}$$

The value of $f_y(x,y)$ at the point $(8,-6)$ is

$$f_y(6,8) = \frac{8}{\sqrt{(6)^2+(8)^2}} = \frac{8}{\sqrt{100}} = \frac{8}{10} = \frac{4}{5} = 0.8$$

(ii) Since $\lim_{x\to\infty} \frac{\ln x}{x} = \frac{\infty}{\infty}$, you can apply L'Hopital's Rule,as follows

$$\lim_{x\to\infty} \frac{lnx}{x} = \lim_{x\to\infty} \frac{\frac{d}{dx}[lnx]}{\frac{d}{dx}[x]} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

2. (i) Since $f(x) = ax^2(2-x)$ is a probability density function on the interval $[0,2]$. So, evaluate the following integral

$$1 = \int_0^2 ax^2(2-x)dx = a\int_0^2 x^2(2-x)dx = a\left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2 = a\left[\frac{16}{3} - 4\right] = \frac{4a}{3}$$

Thus, $a = \frac{3}{4}$.

(ii)

$$\begin{aligned} P\left(0 \le x \le \frac{1}{2}\right) &= \int_0^{1/2} \frac{3}{4}x^2(2-x)dx = \frac{3}{4}\int_0^{1/2} x^2(2-x)dx = \frac{3}{4}\left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^{1/2} \\ &= \frac{3}{4}\left[\frac{1}{12} - \frac{1}{64}\right] = \frac{13}{256} \end{aligned}$$

3. (i) First you interchange the order of integration so that y is the outer variable ,then y will have constant bounds of integration given by $0 \le y \le 1$.Solving for x in the equation $y = \sqrt{x}$ implies that the bounds for x are $0 \le x \le y^2$.Thus

$$\int_0^1 \int_{\sqrt{x}}^1 sin\left(\frac{y^3+1}{2}\right) dydx = \int_0^1 \int_0^{y^2} sin\left(\frac{y^3+1}{2}\right) dxdy = \int_0^1 sin\left(\frac{y^3+1}{2}\right) y^2 dy$$

$$
\begin{aligned}
(u = \frac{y^3+1}{2}, y: 0 \to 1 \quad &\Rightarrow \quad du = \frac{3}{2}y^2dy, u: 1/2 \to 1) \\
&= \int_{1/2}^{1} sin(u)\frac{2}{3}du = \frac{2}{3}\int_{1/2}^{1} sin(u)du = \frac{-2}{3}[cosu]_{1/2}^{1} \\
&= \frac{-2}{3}[cos(1) - cos(1/2)]
\end{aligned}
$$

(ii) By integration by parts, we can written the integral as follows

$$
\int_0^3 xf''(x)dx \quad = \quad [xf'(x)]_0^3 - \int_0^3 f'(x)dx = [xf'(x)]_0^3 - [f(x)]_0^3 = 3f'(3) - f(3) + f(0)
$$

By assumption $f(0) = 5, f(3) = 5$, and $f'(3) = 4$.Thus

$$
\int_0^3 xf''(x)dx = 3(4) - 5 + 5 = 12
$$

4. (i) Let $u = 1 + cos^2t$, $du = -2sintcostdt$,Then the integral

$$
\int \frac{sintcost}{\sqrt{1+cos^2t}}dt = -\frac{1}{2}\int \frac{1}{\sqrt{u}}du = -u^{1/2} + C = -\sqrt{1+cos^2t} + C
$$

(ii)

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^x}{(1+e^x)^3}dx \quad &= \quad \lim_{a\to-\infty}\int_a^0 \frac{e^x}{(1+e^x)^3}dx + \lim_{b\to\infty}\int_0^b \frac{e^x}{(1+e^x)^3}dx \\
(u = 1+e^x, x: a \to 0, x: 0 \to b \quad &\Rightarrow \quad du = e^xdx, u: 1+e^a \to 2, u: 2 \to 1+e^b) \\
&= \lim_{a\to-\infty}\int_{1+e^a}^{2} \frac{1}{u^3}du + \lim_{b\to\infty}\int_2^{1+e^b} \frac{1}{u^3}du \\
&= \lim_{a\to-\infty}\left[-\frac{1}{2u^2}\right]_{1+e^a}^{2} + \lim_{b\to\infty}\left[-\frac{1}{2u^2}\right]_2^{1+e^b} \\
&= \lim_{a\to-\infty}\left[-\frac{1}{8} + \frac{1}{2(1+e^a)^2}\right] + \lim_{b\to\infty}\left[-\frac{1}{2(1+e^b)^2} + \frac{1}{8}\right] \\
&= -\frac{1}{8} + \lim_{a\to-\infty}\frac{1}{2(1+e^a)^2} - \lim_{b\to\infty}\frac{1}{2(1+e^b)^2} + \frac{1}{8} \\
&= -\frac{1}{8} + \frac{1}{2} + \frac{1}{8} = \frac{1}{2}
\end{aligned}
$$

5. (i) $y' = y(1-y) \Rightarrow \frac{dy}{dt} = y(1-y) \Rightarrow \frac{1}{y(1-y)}dy = dt$

Integrate both sides $\int dt = \int \frac{1}{y(1-y)}dy$.Then

$$
t + C_1 = \int \frac{1}{y}dy + \int \frac{1}{1-y}dy = ln \mid y \mid -ln \mid 1-y \mid = ln \mid \frac{y}{1-y} \mid
$$

$$\Rightarrow \quad lnC_2e^t = ln \mid \frac{y}{1-y} \mid \Rightarrow Ce^t = \frac{y}{1-y} \Rightarrow Ce^t - yCe^t = y$$

$$\Rightarrow \quad (1+Ce^t)y = Ce^t \Rightarrow y(t) = \frac{Ce^t}{1+Ce^t}$$

(ii) $\frac{1}{3} = y(0) = \frac{Ce^{(0)}}{1+Ce^{(0)}} = \frac{C}{1+C} \Rightarrow C = \frac{1}{2}$.Thus the solution for the equation with initial condition is $y(t) = \frac{e^t}{2+e^t}$

6. (i) When the ball hits the ground the first time, it has traveled a distance of

$$D_1 = 18$$

.Between the first and second times it hits the ground, it has traveled an additional distance of

$$D_2 = 18\left(\frac{7}{10}\right) + 18\left(\frac{7}{10}\right) = 36\left(\frac{7}{10}\right)$$

Between the second and third times the ball hits the ground, it has traveled an additional distance of

$$D_3 = 18\left(\frac{7}{10}\right)\left(\frac{7}{10}\right) + 18\left(\frac{7}{10}\right)\left(\frac{7}{10}\right) = 36\left(\frac{7}{10}\right)^2$$

Continuing this process,you obtain a total vertical distance traveled of

$$\begin{aligned} D &= 18 + 36\left(\frac{7}{10}\right) + 36\left(\frac{7}{10}\right)^2 + \cdots = -18 + 36 + 36\left(\frac{7}{10}\right) + 36\left(\frac{7}{10}\right)^2 + \cdots \\ &= -18 + \sum_{n=0}^{\infty} 36\left(\frac{7}{10}\right)^n = -18 + \frac{36}{1-\left(\frac{7}{10}\right)} = -18 + 120 = 102 \end{aligned}$$

(ii) The infinite series

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k^2}} = \frac{1}{1^{2/3}} + \frac{1}{2^{2/3}} + \frac{1}{3^{2/3}} + \cdots$$

is a p-series with $p = \frac{2}{3}$.Because $p < 1$,you can conclude that the series divergence.

7. (i) The first derivative of f is $f'(x) = 3x^2$. Thus ,the iterative formula for Newton's Method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x^3 - 66}{3x^2}$$

The calculations for two iterations are shown in the table.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	4	−2	48	0.04167	4.04167
2	4.04167	0.02107	49.00529	0.00043	4.0412
3	4.0412				

Thus,the approximation is $\sqrt[3]{67} = 4.0412$

(ii) Begin by finding several derivatives of f and evaluation each at $c = 64$

$$\begin{aligned} g(x) &= x^{1/3} & g(64) &= 4 \\ g'(x) &= \frac{1}{3}x^{-2/3} & g'(64) &= \frac{1}{3}(0.0625) = 0.02083 \\ g''(x) &= \frac{-2}{9}x^{-5/3} & g''(64) &= \frac{-2}{9}(0.00098) = -0.00011 \end{aligned}$$

Thus, the two-degree Taylor polynomial is

$$g(x) = g(64) + g'(64)(x-64) + \frac{g''(64)(x-64)^2}{2!} = 4 + 0.02083(x-64) - 0.00011(x-64)^2$$

To evaluate the series when $x = 66$.

$$g(66) = 4+0.02083(66-64)-0.00011(66-64)^2 = 4+0.02083(2)-0.00011(4) = 4.04122$$

8. (i)

$$\begin{aligned} E[X] &= 0P(X=0) + 1P(X=1) + 2P(X=2) + 3P(X=3) + \cdots \\ &= e^{-2}\frac{2^1}{1!} + 2e^{-2}\frac{2^2}{2!} + 3e^{-2}\frac{2^3}{3!} + \cdots = 2e^{-2}(1 + e^{-2}\frac{2^1}{1!} + e^{-2}\frac{2^2}{2!} + \cdots) \\ &= 2e^{-2}\sum_{n=0}^{\infty}\frac{2^n}{n!} = 2e^{-2}e^2 = 2 \end{aligned}$$

(ii)

$$\begin{aligned} \mu &= \int_0^{\infty} x\left(5e^{-5x}\right)dx = \lim_{a\to\infty}\int_0^a x\left(5e^{-5x}\right)dx \\ &= \lim_{a\to\infty}\left[-xe^{-5x}\Big|_0^a + \int_0^a e^{-5x}dx\right] = \lim_{a\to\infty}\left[-ae^{-5a} - \frac{1}{5}e^{-5x}\Big|_0^a\right] \\ &= \lim_{a\to\infty}\left[-ae^{-5a} - \frac{1}{5e^{5a}} + \frac{1}{5}\right] = \lim_{a\to\infty}\frac{-a}{e^{-5a}} - \lim_{a\to\infty}\frac{1}{5e^{5a}} + \frac{1}{5} \\ &= -\lim_{a\to\infty}\frac{1}{5e^{5a}} + \frac{1}{5} = \frac{1}{5} \\ \int_0^{\infty} x^2\left(5e^{-5x}\right)dx &= \lim_{a\to\infty}\int_0^a x^2\left(5e^{-5x}\right)dx \end{aligned}$$

$$
\begin{aligned}
&= \lim_{a\to\infty}\left[-x^2e^{-5x}\Big|_0^a - \frac{2}{5}\,xe^{-5x}\Big|_0^a + \frac{2}{5}\int_0^a e^{-5x}dx\right] \\
&= \lim_{a\to\infty}\left[\frac{-a^2}{e^{-5a}} - \frac{2a}{5e^{5a}} - \frac{2}{5}\,e^{-5x}\Big|_0^a\right] \\
&= \lim_{a\to\infty}\frac{-a^2}{e^{5a}} - \lim_{a\to\infty}\frac{2a}{5e^{5a}} - \lim_{a\to\infty}\frac{2}{25e^{5a}} + \frac{2}{25} \\
&= -\lim_{a\to\infty}\frac{2a}{5e^{4a}} - \lim_{a\to\infty}\frac{2}{25e^{5a}} + \frac{2}{25} = -\lim_{a\to\infty}\frac{2}{25e^{5a}} + \frac{2}{25} \\
&= \frac{2}{25}
\end{aligned}
$$

Thus

$$
\sigma = \sqrt{\frac{2}{25} - (\frac{1}{5})^2} = \sqrt{\frac{1}{25}} = \frac{1}{5}
$$

9. Finding several derivatives of f and evaluation each at $c = 0$

$$
\begin{aligned}
f(x) &= ln\left(\tfrac{1+x}{1-x}\right) = ln(1+x) - ln(1-x) && f(0) = 0 \\
f'(x) &= \frac{1}{1+x} - \frac{-1}{1-x} = \frac{2}{1-x^2} && f'(0) = 2 \\
f''(x) &= \frac{0-(-2x)}{\left(1-x^2\right)^2} = \frac{2x}{\left(1-x^2\right)^2} && f''(0) = 0 \\
f^{(3)}(x) &= \frac{2\left(1-x^2\right)^2 - 2x(2(1-x^2)(-2x))}{\left(1-x^2\right)^4} = \frac{2+6x^2}{\left(1-x^2\right)^3} && f^{(3)}(0) = 2
\end{aligned}
$$

Continuing this process,we can see that $f^{(2n-2)}(0) = 0, f^{(2n-1)}(0) = 2$, Thus the Taylor series

$$
\begin{aligned}
f(x) &= f(0) + f'(0)x + \frac{f'(0)x^2}{2!} + \frac{f'(0)x^3}{3!} + \frac{f'(0)x^4}{4!} + \cdots \\
&= 2x + \frac{2}{3!}x^3 + \frac{2}{5!}x^5 + \cdots = \sum_{n=0}^{\infty}\frac{2x^{2n+1}}{(2n+1)!}
\end{aligned}
$$

For this power series, $a_n = \dfrac{2}{(2n+1)!}$

$$
\lim_{n\to\infty}\left|\frac{a_{n+1}x^{2(n+1)+1}}{a_nx^{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{2}{(2n+3)!}x^{2n+3}}{\frac{2}{(2n+1)!}x^{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{x^2}{(2n+3)(2n+2)}\right| = 0
$$

So,by the Ration Test, this series converges for all x and the radius of convergence is ∞

10. Let $T(x,y,z)=8x^2yz$ and $g(x,y,z)=x^2+y^2+z^2-1$.Then, define a new function $F(x,y,x,\lambda)$ by

$$F(x,y,x,\lambda)=T(x,y,z)-\lambda(x,y,z)=8x^2yz-\lambda\left(x^2+y^2+z^2-1\right)$$

To find the critical numbers of F, set the partial derivatives of F with respect to x,y,z,and λ equal to zero and obtain

$$F_x(x,y,x,\lambda)=16xyz-2\lambda x=0, \qquad F_y(x,y,x,\lambda)=8x^2z-2\lambda y=0,$$
$$F_z(x,y,x,\lambda)=8x^2y-2\lambda z=0 \qquad ,F_\lambda(x,y,x,\lambda)=-x^2-y^2-z^2+1=0$$

Then

$$\frac{F_x}{F_y}:\frac{2y}{x}=\frac{x}{y},\frac{F_y}{F_z}:\frac{z}{y}=\frac{y}{z},\frac{F_x}{F_z}:\frac{2z}{x}=\frac{x}{z}$$
$$\Rightarrow \quad x^2=2y^2,z^2=y^2$$

Substitute this into the equation $F_\lambda(x,y,x,\lambda)=-x^2-y^2-z^2+1=0$ and solve y

$$0=F_\lambda(x,y,x,\lambda)=-2y^2-y^2-y^2+1\Rightarrow y^2=\frac{1}{4}=\pm\frac{1}{2}$$

Using this y-value,you can conclude that the critical values are

$$y=\frac{1}{2}\Rightarrow x=\pm\frac{1}{\sqrt{2}},z=\pm\frac{1}{2}$$
$$x=\frac{-1}{2}\Rightarrow x=\pm\frac{1}{\sqrt{2}},z=\pm\frac{1}{2}$$

which implies that the temperature value is

$$T\left(\pm\frac{1}{\sqrt{2}},\frac{1}{2},\frac{1}{2}\right)=8(1/2)(1/2)(1/2)=1,T\left(\pm\frac{1}{\sqrt{2}},-\frac{1}{2},\frac{1}{2}\right)=8(1/2)(-1/2)(1/2)=-1,$$
$$T\left(\pm\frac{1}{\sqrt{2}},\frac{1}{2},-\frac{1}{2}\right)=8(1/2)(1/2)(-1/2)=-1,T\left(\pm\frac{1}{\sqrt{2}},-\frac{1}{2},-\frac{1}{2}\right)=8(1/2)(1/2)(1/2)=1,$$

Thus,the point(s) on the sphere at which the temperature is greatest is

$$\left(\pm\frac{1}{\sqrt{2}},\frac{1}{2},\frac{1}{2}\right),\left(\pm\frac{1}{\sqrt{2}},-\frac{1}{2},-\frac{1}{2}\right)$$

the point(s) on the sphere at which the temperature is least is

$$\left(\pm\frac{1}{\sqrt{2}},-\frac{1}{2},\frac{1}{2}\right),\left(\pm\frac{1}{\sqrt{2}},\frac{1}{2},-\frac{1}{2}\right)$$