CALCULUS Final SOLUTION

Exam Set: A

1. (i) To find the first partial derivative with respect to y,hold x constant to obtain

$$f_y(x,y) = \frac{\partial}{\partial y}\sqrt{x^2+y^2} = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}$$

The value of $f_y(x,y)$ at the point $(8,-6)$ is

$$f_y(8,-6) = \frac{-6}{\sqrt{(8)^2+(-6)^2}} = \frac{-6}{\sqrt{100}} = \frac{-6}{10} = -\frac{3}{5} = -0.6$$

(ii) Since $\lim_{x\to\infty}\frac{x}{\ln x} = \frac{\infty}{\infty}$, you can apply L'Hopital's Rule,as follows

$$\lim_{x\to\infty}\frac{x}{lnx} = \lim_{x\to\infty}\frac{\frac{d}{dx}[x]}{\frac{d}{dx}[\ln x]} = \lim_{x\to\infty}\frac{1}{\frac{1}{x}} = \lim_{x\to\infty} x = \infty \text{ div}$$

2. (i) Since $f(x) = ax^2(1-x)$ is a probability density function on the interval $[0,1]$. So, evaluate the following integral

$$1 = \int_0^1 ax^2(1-x)dx = a\int_0^1 x^2(1-x)dx = a\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = a\left[\frac{1}{3}-\frac{1}{4}\right] = \frac{a}{12}$$

Thus, $a = 12$.

(ii)

$$\begin{aligned} P\left(0 \le x \le \frac{1}{2}\right) &= \int_0^{1/2} 12x^2(1-x)dx = 12\int_0^{1/2} x^2(1-x)dx = 12\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^{1/2} \\ &= 4\left[x^3\right]_0^{1/2} - 3\left[x^4\right]_0^{1/2} = 4\left(\frac{1}{8}\right) - 3\left(\frac{1}{16}\right) = \frac{5}{16} = 0.3125 \end{aligned}$$

3. (i) First you interchange the order of integration so that y is the outer variable ,then y will have constant bounds of integration given by $0 \le y \le 1$.Solving for x in the equation $y = \sqrt{x}$ implies that the bounds for x are $0 \le x \le y^2$.Thus

$$\int_0^1\int_{\sqrt{x}}^1 sin\left(\frac{y^3+1}{2}\right)dydx \quad = \quad \int_0^1\int_0^{y^2} sin\left(\frac{y^3+1}{2}\right)dxdy = \int_0^1 sin\left(\frac{y^3+1}{2}\right)y^2dy$$

$$\begin{aligned}(u = \frac{y^3+1}{2}, y: 0 \to 1 \quad &\Rightarrow \quad du = \frac{3}{2}y^2 dy, u: 1/2 \to 1)\\ &= \int_{1/2}^{1} sin(u)\frac{2}{3}du = \frac{2}{3}\int_{1/2}^{1} sin(u)du = \frac{-2}{3}[cosu]_{1/2}^{1}\\ &= \frac{-2}{3}[cos(1) - cos(1/2)]\end{aligned}$$

(ii) By integration by parts, we can written the integral as follows

$$\int_0^3 xf''(x)dx \quad = \quad [xf'(x)]_0^3 - \int_0^3 f'(x)dx = [xf'(x)]_0^3 - [f(x)]_0^3 = 3f'(0) - f(3) + f(0)$$

By assumption $f(0) = 4, f(3) = 5$, and $f'(3) = 5$.Thus

$$\int_0^3 xf''(x)dx = 3(5) - 5 + 4 = 14$$

4. (i) Let $u = 1 + cos^2 t$, $du = -2sintcostdt$,Then the integral

$$\int \frac{sintcost}{\sqrt{1+cos^2 t}}dt = -\frac{1}{2}\int \frac{1}{\sqrt{u}}du = -u^{1/2} + C = -\sqrt{1+cos^2 t} + C$$

(ii)

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^x}{(1+e^x)^2}dx \quad &= \quad \lim_{a\to-\infty}\int_a^0 \frac{e^x}{(1+e^x)^2}dx + \lim_{b\to\infty}\int_0^b \frac{e^x}{(1+e^x)^2}dx\\ (u = 1+e^x, x: a \to 0, x: 0 \to b \quad &\Rightarrow \quad du = e^x dx, u: 1+e^a \to 2, u: 2 \to 1+e^b)\\ &= \lim_{a\to-\infty}\int_{1+e^a}^{2} \frac{1}{u^2}du + \lim_{b\to\infty}\int_2^{1+e^b} \frac{1}{u^2}du\\ &= \lim_{a\to-\infty}\left[-\frac{1}{u}\right]_{1+e^a}^{2} + \lim_{b\to\infty}\left[-\frac{1}{u}\right]_2^{1+e^b}\\ &= \lim_{a\to-\infty}\left[-\frac{1}{2} + \frac{1}{1+e^a}\right] + \lim_{b\to\infty}\left[-\frac{1}{1+e^b} + \frac{1}{2}\right]\\ &= -\frac{1}{2} + \lim_{a\to-\infty}\frac{1}{1+e^a} - \lim_{b\to\infty}\frac{1}{1+e^b} + \frac{1}{2}\\ &= -\frac{1}{2} + 1 + \frac{1}{2} = 1\end{aligned}$$

5. (i) $y' = y(1-y) \Rightarrow \frac{dy}{dt} = y(1-y) \Rightarrow \frac{1}{y(1-y)}dy = dt$

Integrate both sides $\int dt = \int \frac{1}{y(1-y)}dy$.Then

$$t + C_1 = \int \frac{1}{y}dy + \int \frac{1}{1-y}dy = ln \mid y \mid -ln \mid 1-y \mid = ln \mid \frac{y}{1-y} \mid$$

$$\Rightarrow \quad lnC_2e^t = ln \mid \frac{y}{1-y} \mid \Rightarrow Ce^t = \frac{y}{1-y} \Rightarrow Ce^t - yCe^t = y$$

$$\Rightarrow \quad (1+Ce^t)y = Ce^t \Rightarrow y(t) = \frac{Ce^t}{1+Ce^t}$$

(ii) $\frac{1}{2} = y(0) = \frac{Ce^{(0)}}{1+Ce^{(0)}} = \frac{C}{1+C} \Rightarrow C = 1$.Thus the solution for the equation with initial condition is $y(t) = \frac{e^t}{1+e^t}$

6. (i) When the ball hits the ground the first time, it has traveled a distance of

$$D_1 = 16$$

.Between the first and second times it hits the ground, it has traveled an additional distance of

$$D_2 = 16\left(\frac{3}{5}\right) + 16\left(\frac{3}{5}\right) = 32\left(\frac{3}{5}\right)$$

Between the second and third times the ball hits the ground, it has traveled an additional distance of

$$D_3 = 16\left(\frac{3}{5}\right)\left(\frac{3}{5}\right) + 16\left(\frac{3}{5}\right)\left(\frac{3}{5}\right) = 32\left(\frac{3}{5}\right)^2$$

Continuing this process,you obtain a total vertical distance traveled of

$$\begin{aligned} D &= 16 + 32\left(\frac{3}{5}\right) + 32\left(\frac{3}{5}\right)^2 + \cdots = -16 + 32 + 32\left(\frac{3}{5}\right) + 32\left(\frac{3}{5}\right)^2 + \cdots \\ &= -16 + \sum_{n=0}^{\infty} 32\left(\frac{3}{5}\right)^n = -16 + \frac{32}{1-\left(\frac{3}{5}\right)} = -16 + 80 = 64 \end{aligned}$$

(ii) The infinite series

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[2]{k^3}} = \frac{1}{1^{3/2}} + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \cdots$$

is a p-series with $p = \frac{3}{2}$.Because $p > 1$,you can conclude that the series converges.

7. (i) The first derivative of f is $f'(x) = 3x^2$. Thus ,the iterative formula for Newton's Method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x^3 - 67}{3x^2}$$

The calculations for two iterations are shown in the table.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	4	-3	48	-0.0625	4.0625
2	4.0625	0.04712	49.51172	0.00095	4.06155
3	4.06115				

Thus,the approximation is $\sqrt[3]{67} = 4.06115$

(ii) Begin by finding several derivatives of f and evaluation each at $c = 64$

$$\begin{aligned} g(x) &= x^{1/3} & g(64) &= 4 \\ g'(x) &= \frac{1}{3}x^{-2/3} & g'(64) &= \frac{1}{3}(0.0625) = 0.02083 \\ g''(x) &= \frac{-2}{9}x^{-5/3} & g''(64) &= \frac{-2}{9}(0.00098) = -0.00011 \end{aligned}$$

Thus, the two-degree Taylor polynomial is

$$g(x) = g(64) + g'(64)(x-64) + \frac{g''(64)(x-64)^2}{2!} = 4 + 0.02083(x-64) - 0.00011(x-64)^2$$

To evaluate the series when $x = 67$.

$$g(67) = 4+0.02083(67-64)-0.00011(67-64)^2 = 4+0.02083(3)-0.00011(9) = 4.0615$$

8. (i)

$$\begin{aligned} E[X] &= 0P(X=0) + 1P(X=1) + 2P(X=2) + 3P(X=3) + \cdots \\ &= e^{-3}\frac{3^1}{1!} + 2e^{-3}\frac{3^2}{2!} + 3e^{-3}\frac{3^3}{3!} + \cdots = 3e^{-3}(1 + e^{-3}\frac{3^1}{1!} + e^{-3}\frac{3^2}{2!} + \cdots) \\ &= 3e^{-3}\sum_{n=0}^{\infty}\frac{3^n}{n!} = 3e^{-3}e^3 = 3 \end{aligned}$$

(ii)

$$\begin{aligned} \mu &= \int_0^{\infty} x\left(4e^{-4x}\right)dx = \lim_{a\to\infty}\int_0^a x\left(4e^{-4x}\right)dx \\ &= \lim_{a\to\infty}\left[-xe^{-4x}\Big|_0^a + \int_0^a e^{-4x}dx\right] = \lim_{a\to\infty}\left[-ae^{-4a} - \frac{1}{4}e^{-4x}\Big|_0^a\right] \\ &= \lim_{a\to\infty}\left[-ae^{-4a} - \frac{1}{4e^{4a}} + \frac{1}{4}\right] = \lim_{a\to\infty}\frac{-a}{e^{-4a}} - \lim_{a\to\infty}\frac{1}{4e^{4a}} + \frac{1}{4} \\ &= -\lim_{a\to\infty}\frac{1}{4e^{4a}} + \frac{1}{4} = \frac{1}{4} \\ \int_0^{\infty} x^2\left(4e^{-4x}\right)dx &= \lim_{a\to\infty}\int_0^a x^2\left(4e^{-4x}\right)dx \end{aligned}$$

$$
\begin{aligned}
&= \lim_{a\to\infty}\left[-x^2e^{-4x}\big|_0^a - \frac{1}{2}\ xe^{-4x}\big|_0^a + \frac{1}{2}\int_0^a e^{-4x}dx\right] \\
&= \lim_{a\to\infty}\left[\frac{-a^2}{e^{-4a}} - \frac{a}{2e^{4a}} - \frac{1}{8}\ e^{-4x}\big|_0^a\right] \\
&= \lim_{a\to\infty}\frac{-a^2}{e^{4a}} - \lim_{a\to\infty}\frac{a}{2e^{4a}} - \lim_{a\to\infty}\frac{1}{8e^{4a}} + \frac{1}{8} \\
&= \lim_{a\to\infty}\frac{-2a}{4e^{4a}} - \lim_{a\to\infty}\frac{1}{8e^{4a}} + \frac{1}{8} = \lim_{a\to\infty}\frac{-2}{16e^{4a}} + \frac{1}{8} \\
&= \frac{1}{8}
\end{aligned}
$$

Thus

$$
\sigma = \sqrt{\frac{1}{8} - (\frac{1}{4})^2} = \sqrt{\frac{1}{16}} = \frac{1}{4}
$$

9. Finding several derivatives of f and evaluation each at $c = 0$

$$
\begin{aligned}
f(x) &= ln\left(\tfrac{1+x}{1-x}\right) = ln(1+x) - ln(1-x) && f(0) = 0 \\
f'(x) &= \frac{1}{1+x} - \frac{-1}{1-x} = \frac{2}{1-x^2} && f'(0) = 2 \\
f''(x) &= \frac{0-(-2x)}{\left(1-x^2\right)^2} = \frac{2x}{\left(1-x^2\right)^2} && f''(0) = 0 \\
f^{(3)}(x) &= \frac{2\left(1-x^2\right)^2 - 2x(2(1-x^2)(-2x))}{\left(1-x^2\right)^4} = \frac{2+6x^2}{\left(1-x^2\right)^3} && f^{(3)}(0) = 2
\end{aligned}
$$

Continuing this process,we can see that $f^{(2n-2)}(0) = 0, f^{(2n-1)}(0) = 2$, Thus the Taylor series

$$
\begin{aligned}
f(x) &= f(0) + f'(0)x + \frac{f'(0)x^2}{2!} + \frac{f'(0)x^3}{3!} + \frac{f'(0)x^4}{4!} + \cdots \\
&= 2x + \frac{2}{3!}x^3 + \frac{2}{5!}x^5 + \cdots = \sum_{n=0}^{\infty}\frac{2x^{2n+1}}{(2n+1)!}
\end{aligned}
$$

For this power series, $a_n = \dfrac{2}{(2n+1)!}$

$$
\lim_{n\to\infty}\left|\frac{a_{n+1}x^{2(n+1)+1}}{a_nx^{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{2}{(2n+3)!}x^{2n+3}}{\frac{2}{(2n+1)!}x^{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{x^2}{(2n+3)(2n+2)}\right| = 0
$$

So,by the Ration Test, this series converges for all x and the radius of convergence is ∞

10. Let $T(x,y,z) = 10xy^2z$ and $g(x,y,z) = x^2+y^2+z^2-1$.Then, define a new function $F(x,y,x,\lambda)$ by

$$F(x,y,x,\lambda) = T(x,y,z) - \lambda(x,y,z) = 10xy^2z - \lambda\left(x^2+y^2+z^2-1\right)$$

To find the critical numbers of F, set the partial derivatives of F with respect to x,y,z,and λ equal to zero and obtain

$$F_x(x,y,x,\lambda) = 10y^2z - 2\lambda x = 0, \qquad F_y(x,y,x,\lambda) = 20xyz - 2\lambda y = 0,$$
$$F_z(x,y,x,\lambda) = 10xy^2 - 2\lambda z = 0 \qquad , F_\lambda(x,y,x,\lambda) = -x^2 - y^2 - z^2 + 1 = 0$$

Then

$$\frac{F_x}{F_y} : \frac{y}{2x} = \frac{x}{y}, \frac{F_y}{F_z} : \frac{2z}{y} = \frac{y}{z}, \frac{F_x}{F_z} : \frac{x}{z} = \frac{z}{x}$$
$$\Rightarrow \quad y^2 = 2x^2, z^2 = x^2$$

Substitute this into the equation $F_\lambda(x,y,x,\lambda) = -x^2 - y^2 - z^2 + 1 = 0$ and solve x

$$0 = F_\lambda(x,y,x,\lambda) == -x^2 - 2x^2 - x^2 + 1 \Rightarrow x^2 = \frac{1}{4} = \pm\frac{1}{2}$$

Using this x-value,you can conclude that the critical values are

$$x = \frac{1}{2} \Rightarrow y = \pm\frac{1}{\sqrt{2}}, z = \pm\frac{1}{2}$$

$$x = \frac{-1}{2} \Rightarrow y = \pm\frac{1}{\sqrt{2}}, z = \pm\frac{1}{2}$$

which implies that the temperature value is

$$T\left(\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right) = 10(1/2)(1/2)(1/2) = \frac{5}{4}, T\left(\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, -\frac{1}{2}\right) = 10(1/2)(1/2)(-1/2) = -\frac{5}{4}$$
$$T\left(-\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right) = 10(-1/2)(1/2)(1/2) = -\frac{5}{4}, T\left(-\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, -\frac{1}{2}\right) = 10(-1/2)(1/2)(-1/2) = \frac{5}{4}$$

Thus,the point(s) on the sphere at which the temperature is greatest is

$$\left(\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right), \left(-\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$$

the point(s) on the sphere at which the temperature is least is

$$\left(\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, -\frac{1}{2}\right), \left(-\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$$