FINAL OF CALCULUS

Exam Set: **D**

No credit will be given for an answer without reasoning.

1.

(i) [5%] Evaluate the integral

$$\int \frac{\sin t \cos t}{\sqrt{1+\cos^2 t}}\, dt.$$

(ii) [5%] Find the improper integral

$$\int_{-\infty}^{\infty} \frac{e^x}{(1+e^x)^3}\, dx.$$

2.

(i) [5%] Find $f_y(6,8)$ for $f(x,y)=\sqrt{x^2+y^2}$.

(ii) [5%] Find the limit

$$\lim_{x\to\infty} \frac{\ln x}{x}.$$

3.

(i) [5%] Evaluate the double integral by changing the order of integration:

$$\int_0^1 \int_{\sqrt{x}}^1 \sin\left(\frac{y^3+1}{2}\right) dy\, dx.$$

(ii) [5%] Let f be a differentiable function such that $f(0)=5$, $f(3)=5$, and $f'(3)=4$. What is $\int_0^3 x f''(x)\, dx$?

4. Consider the differential equation: $y'=y(1-y)$.

(i) [5%] Find the general solution $y(t)$.

(ii) [5%] Find the solution for the equation with initial condition $y(0)=1/3$.

5.

(i) [5%] Consider a random variable X with probability

$$P(X=k)=e^{-2}\frac{2^k}{k!}$$

for $k=0,1,2,3,\ldots$. Find the expected value $E[X]$.

(ii) [5%] Find the standard deviation σ of the probability density function $f(x)=5e^{-5x}$, $0\le x<\infty$.

6.

(i) [5%] Find the constant a such that the function $f(x)=ax^2(2-x)$ is a probability density function on the interval $[0,2]$.

(ii) [5%] Keep the assumption in (i). Find the probability $P(0\le x\le 1/2)$.

7.

(i) [5%] A ball is dropped from a height of 18 meters. Each time it drops h meters, it rebounds $0.7h$ meters. Find the total vertical distance travelled by the ball.

(ii) [5%] Determine the convergence or divergence of the series

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k^2}}.$$

8. Estimate $\sqrt[3]{66}$ by the following two methods:

(i) [5%] Apply Newton's method for two iterations to the function $f(x) = x^3 - 66$ with initial guess $x_1 = 4$.

(ii) [5%] Use the degree-two Taylor polynomial centered at $x = 64$ to approximate the function $g(x) = x^{1/3}$. Then evaluate the Taylor polynomial at $x = 66$.

9. [10%] Assume that the temperature T at a point (x, y, z) on the sphere $x^2 + y^2 + z^2 = 1$ is given by

$$T(x, y, z) = 8x^2yz.$$

Use Lagrange multiplier method to find the point(s) on the sphere at which the temperature is greatest and the point(s) at which it is least.

10. [10%] Find the Taylor series centered at $x = 0$ of the function

$$f(x) = \ln\left(\frac{1+x}{1-x}\right).$$

Then find the radius of convergence of the Taylor series. [Hint: $f(x) = \ln(1 + x) - \ln(1 - x)$]