

FINAL OF CALCULUS

Exam Set: **C**

No credit will be given for an answer without reasoning.

1.

- (i) [5%] Find the constant a such that the function $f(x) = ax^2(1 - x)$ is a probability density function on the interval $[0, 1]$.
- (ii) [5%] Keep the assumption in (i). Find the probability $P(0 \leq x \leq 1/2)$.

2.

- (i) [5%] Find $f_y(8, -6)$ for $f(x, y) = \sqrt{x^2 + y^2}$.
- (ii) [5%] Find the limit

$$\lim_{x \rightarrow \infty} \frac{x}{\ln x}.$$

3.

- (i) [5%] A ball is dropped from a height of 16 meters. Each time it drops h meters, it rebounds $0.6h$ meters. Find the total vertical distance travelled by the ball.
- (ii) [5%] Determine the convergence or divergence of the series

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[2]{k^3}}.$$

4. Consider the differential equation: $y' = y(1 - y)$.

- (i) [5%] Find the general solution $y(t)$.
- (ii) [5%] Find the solution for the equation with initial condition $y(0) = 1/2$.

5.

- (i) [5%] Evaluate the integral

$$\int \frac{\sin t \cos t}{\sqrt{1 + \cos^2 t}} dt.$$

- (ii) [5%] Find the improper integral

$$\int_{-\infty}^{\infty} \frac{e^x}{(1 + e^x)^2} dx.$$

6.

- (i) [5%] Evaluate the double integral by changing the order of integration:

$$\int_0^1 \int_{\sqrt{x}}^1 \sin\left(\frac{y^3 + 1}{2}\right) dy dx.$$

- (ii) [5%] Let f be a differentiable function such that $f(0) = 4$, $f(3) = 5$, and $f'(3) = 5$. What is $\int_0^3 x f''(x) dx$?

7. [10%] Assume that the temperature T at a point (x, y, z) on the sphere $x^2 + y^2 + z^2 = 1$ is given by

$$T(x, y, z) = 8xy^2z.$$

Use Lagrange multiplier method to find the point(s) on the sphere at which the temperature is greatest and the point(s) at which it is least.

8. Estimate $\sqrt[3]{67}$ by the following two methods:

- (i) [5%] Apply Newton's method for two iterations to the function $f(x) = x^3 - 67$ with initial guess $x_1 = 4$.
- (ii) [5%] Use the degree-two Taylor polynomial centered at $x = 64$ to approximate the function $g(x) = x^{1/3}$. Then evaluate the Taylor polynomial at $x = 67$.

9.

- (i) [5%] Consider a random variable X with probability

$$P(X = k) = e^{-4} \frac{4^k}{k!}$$

for $k = 0, 1, 2, 3, \dots$. Find the expected value $E[X]$.

- (ii) [5%] Find the standard deviation σ of the probability density function $f(x) = 4e^{-4x}$, $0 \leq x < \infty$.

10. [10%] Find the Taylor series centered at $x = 0$ of the function

$$f(x) = \ln \left(\frac{1+x}{1-x} \right).$$

Then find the radius of convergence of the Taylor series. [Hint: $f(x) = \ln(1+x) - \ln(1-x)$]