

Calculus Examination2 Solution

TEST C

1. $f_x = -e^{\cos x} y \sin x$

$$f_y = e^{\cos x}$$

$$f_{xx} = -e^{\cos x} y \cos x + e^{\cos x} y \sin^2 x$$

$$f_{xy} = f_{yx} = -e^{\cos x} \sin x$$

$$f_{yy} = 0$$

2. $f(x) = \frac{\cos x}{1+\sin^2 x}$

$$f'(x) = -\frac{\sin x}{1+\sin^2 x} - \frac{2 \cos^2 x \sin x}{(1+\sin^2 x)^2} = \frac{-\sin x(1+\sin^2 x+2 \cos^2 x)}{(1+\sin^2 x)^2} = \frac{-\sin x(2+\cos^2 x)}{(1+\sin^2 x)^2}$$

$$f'(x) = 0 \Rightarrow \sin x = 0 \Rightarrow x = k\pi, k = 0, \pm 1, \pm 2, \pm 3, \dots$$

$$f(0 + 2k\pi) = 1 \text{ is the relative maximum}$$

$$f(\pi + 2k\pi) = -1 \text{ is the relative minimum}$$

3. Let $f(x) = e^{\frac{x}{2}} \tan(2x)$

$$f'(x) = \frac{1}{2}e^{\frac{x}{2}} \tan(2x) + 2e^{\frac{x}{2}} \sec^2(2x)$$

$$f'(0) = 2$$

$$\Rightarrow y = 2x \text{ is the tangent line at point } (0,0)$$

4. $\int_0^4 \int_0^4 (xy)^2 dx dy = \int_0^4 \frac{x^3}{3} y^2 \Big|_0^4 dy = \frac{64}{3} \cdot \frac{y^3}{3} \Big|_0^4 = \frac{4096}{9}$

5. (a) $a = \frac{4 \times (100 \times 34 + 150 \times 44 + 200 \times 52 + 250 \times 63) - (100 + 150 + 200 + 250) \times (34 + 44 + 52 + 63)}{(100^2 + 150^2 + 200^2 + 250^2) - (100 + 150 + 200 + 250)^2} = 0.19$

$$b = \frac{1}{4} \cdot (34 + 44 + 52 + 63) - a(100 + 150 + 200 + 250) = 15$$

$$\text{hence the least square regression line is } y = 0.19x + 15$$

(b) $0.19 \times 160 + 15 = 45.4$

$$6. \quad (a) \int_0^{12} 14 + 8 \sin \frac{\pi(t-8)}{6} dt = 14t - \frac{48}{\pi} \cos \frac{\pi(t-8)}{6} \Big|_0^{12} = 168 - \frac{8}{\pi}(-\frac{1}{2}) + \frac{8}{\pi}(-\frac{1}{2}) = 168$$

$$\begin{aligned} (b) \quad & \frac{1}{12-6} \int_6^{12} 14 + 8 \sin \frac{\pi(t-8)}{6} dt \\ &= \frac{1}{6} (14t - \frac{48}{\pi} \cos \frac{\pi(t-8)}{6}) \Big|_6^{12} \\ &= 14 - \frac{8}{\pi}(-\frac{1}{2}) + \frac{8}{\pi}(\frac{1}{2}) \\ &= 14 + \frac{8}{\pi} \end{aligned}$$

$$7. \quad (a) \int x \cos x dx = \int x d \sin x = x \sin x - \int \sin x dx = x \sin x + \cos x + c$$

$$\begin{aligned} (b) \quad & \int_0^3 \int_0^{x^2} \sqrt{1+x} dy dx \\ &= \int_0^3 y \sqrt{1+x} \Big|_0^{x^2} dx \\ &= \int_0^3 x^2 \sqrt{1+x} dx \\ &= \int_0^3 (x+1-1)^2 \sqrt{x+1} d(x+1) \\ &= \int_0^3 (x+1)^{\frac{5}{2}} - 2(x+1)^{\frac{3}{2}} + (x+1)^{\frac{1}{2}} d(x+1) \\ &= \frac{2}{7}(x+1)^{\frac{7}{2}} - \frac{4}{5}(x+1)^{\frac{5}{2}} + \frac{2}{3}(x+1)^{\frac{3}{2}} \Big|_0^3 \\ &= \frac{1696}{105} \end{aligned}$$

$$(c) \int \frac{\cos^2(\frac{x}{2})}{x+\sin x} dx = \frac{1}{2} \int \frac{1+\cos x}{x+\sin x} dx = \frac{1}{2} \ln |x + \sin x| + c$$

$$\begin{aligned} (d) \quad & \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2x + \cos x)^2 dx \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4x^2 + 4x \cos x + \cos^2 x dx \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4x^2 + 4x \cos x + \frac{\cos 2x+1}{2} dx \\ &= \frac{4}{3}x^3 + 4x \sin x + 4 \cos x + \frac{1}{4} \sin 2x + \frac{x}{2} \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \frac{\pi}{2} + \frac{\pi^3}{3} \end{aligned}$$

$$\begin{aligned} (e) \quad & \int_0^{\pi} \frac{1}{\cos x} dx \\ &= \int_0^{\pi} \sec x dx \\ &= \lim_{a \rightarrow \frac{\pi}{2}^-} \int_0^a \sec x dx + \lim_{b \rightarrow \frac{\pi}{2}^+} \int_b^{\pi} \sec x dx \\ &= \lim_{a \rightarrow \frac{\pi}{2}^-} \int_0^a \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x} dx + \lim_{b \rightarrow \frac{\pi}{2}^+} \int_b^{\pi} \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x} dx \end{aligned}$$

$$\begin{aligned}
&= \lim_{a \rightarrow \frac{\pi}{2}^-} \int_0^a \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx + \lim_{b \rightarrow \frac{\pi}{2}^+} \int_b^\pi \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \\
&= \lim_{a \rightarrow \frac{\pi}{2}^-} \ln |\sec x + \tan x| \Big|_0^a + \lim_{b \rightarrow \frac{\pi}{2}^+} \ln |\sec x + \tan x| \Big|_b^\pi \\
&= \lim_{a \rightarrow \frac{\pi}{2}^-} \ln |\sec a + \tan a| - \lim_{b \rightarrow \frac{\pi}{2}^+} \ln |\sec b + \tan b| \\
&\text{since } \lim_{a \rightarrow \frac{\pi}{2}^-} \ln |\sec a + \tan a| \text{ and } \lim_{b \rightarrow \frac{\pi}{2}^+} \ln |\sec b + \tan b| \text{ are divergent} \\
&\text{hence } \int_0^\pi \frac{1}{\cos x} dx \text{ is divergent}
\end{aligned}$$

8. (a) $\lim_{x \rightarrow 1} \frac{x^2-1}{x-1} = \lim_{x \rightarrow 1} \frac{(x+1)(x-1)}{x-1} = \lim_{x \rightarrow 1} x + 1 = 2$

(b) $\lim_{x \rightarrow \infty} \frac{x^3-x^2+4x-4}{x^3-6x^2+5x} = \lim_{x \rightarrow \infty} \frac{1-\frac{1}{x}+\frac{4}{x^2}-\frac{4}{x^3}}{1-\frac{6}{x}+\frac{5}{x^2}} = 1$

(c) $\lim_{x \rightarrow 0} \frac{\sin(2x)}{5x}$ ($\frac{0}{0}$) apply l'Hopital's rule)
 $= \lim_{x \rightarrow 0} \frac{2 \cos(2x)}{5}$
 $= \frac{2}{5}$

(d) $\lim_{x \rightarrow 0^+} \frac{(\ln x)^4}{x} = \infty$