Calculus Examination2 Solution

TEST B

1. $f_x = e^{\sin y}$

   $f_y = e^{\sin y} x \cos y$

   $f_{xx} = 0$

   $f_{xy} = f_{yx} = e^{\sin y} \cos y$

   $f_{yy} = e^{\sin y} x \cos^2 y - e^{\sin y} x \sin y$

2. $f(x) = \frac{\sin x}{1+3\cos^2 x}$

   $f'(x) = \frac{\cos x}{1+3\cos^2 x} + \frac{6\cos x \sin^2 x}{(1+3\cos^2 x)^2} = \frac{\cos x(1+3\cos^2 x+6\sin^2 x)}{(1+3\cos^2 x)^2} = \frac{\cos x(7-3\cos^2 x)}{(1+3\cos^2 x)^2}$

   $f'(x) = 0 \Rightarrow \cos x = 0 \Rightarrow x = \frac{k\pi}{2} \quad , k = 0, \pm 1, \pm 2, \pm 3, \ldots$

   $f(\frac{\pi}{2} + 2k\pi) = 1$ is the relative maximum

   $f(-\frac{\pi}{2} + 2k\pi) = -1$ is the relative minimum

3. Let f(x) $= e^{\frac{x}{3}} \tan 3x$

   f'(x) $= 3e^{\frac{x}{3}} \sec^2 3x + \frac{1}{3} e^{\frac{x}{3}} \tan(3x)$

   f'(0) $= 3$

   $\Rightarrow y = 3x$ is the tangent line at point (0,0)

4. $\int_0^4 \int_0^4 (xy)^2 \, dx \, dy = \int_0^4 \frac{x^3}{3} y^2 \, |_0^4 \, dy = \frac{64}{3} \cdot \frac{y^3}{3} \, |_0^4 = \frac{4096}{9}$

5. (a) $a = \frac{4\times(100\times36+150\times45+200\times50+250\times56)-(100+150+200+250)\times(36+45+50+56)}{(100^2+150^2+200^2+250^2)-(100+150+200+250)^2} = 0.13$

   $b = \frac{1}{4} \cdot (36 + 45 + 50 + 56) - a(100 + 150 + 200 + 250) = 24$

   hence the least square regression line is $y = 0.13x + 24$

   (b) $0.13 \times 160 + 24 = 44.8$

6. (a) $\int_0^{12}\ 17+7\sin\frac{\pi(t-8)}{6}\ dt = 17t-\frac{42}{\pi}\cos\frac{\pi(t-8)}{6}\ |_0^{12} = 204-\frac{42}{\pi}(-\frac{1}{2})+\frac{42}{\pi}(-\frac{1}{2}) = 204$

   (b) $\frac{1}{12-6}\int_6^{12}\ 17+7\sin\frac{\pi(t-8)}{6}\ dt$

   $=\frac{1}{6}(17t-\frac{42}{\pi}\cos\frac{\pi(t-8)}{6})\ |_6^{12}$

   $=17-\frac{7}{\pi}(-\frac{1}{2})+\frac{7}{\pi}(\frac{1}{2})$

   $=17+\frac{7}{\pi}$

7. (a) $\int\ x\sin x\ dx = \int\ -x\ d\cos x = -x\cos x+\int\ \cos x\ dx = -x\cos x+\sin x+c$

   (b) $\int_0^3\ \int_0^{x^2}\ \sqrt{1+x}\ dy\ dx$

   $=\int_0^3\ y\sqrt{1+x}\ |_0^{x^2}\ dx$

   $=\int_0^3\ x^2\sqrt{1+x}\ dx$

   $=\int_0^3\ (x+1-1)^2\sqrt{x+1}\ d(x+1)$

   $=\int_0^3\ (x+1)^{\frac{5}{2}}-2(x+1)^{\frac{3}{2}}+(x+1)^{\frac{1}{2}}\ d(x+1)$

   $=\frac{2}{7}(x+1)^{\frac{7}{2}}-\frac{4}{5}(x+1)^{\frac{5}{2}}+\frac{2}{3}(x+1)^{\frac{3}{2}}\ |_0^3$

   $=\frac{1696}{105}$

   (c) $\int\ \frac{1+2\sin^2(\frac{x}{2})}{2x-\sin x}\ dx = \int\ \frac{2-\cos x}{2x-\sin x}\ dx = \ln|2x-\sin x|+c$

   (d) $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\ (2x+\cos x)^2\ dx$

   $=\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\ 4x^2+4x\cos x+\cos^2 x\ dx$

   $=\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\ 4x^2+4x\cos x+\frac{\cos 2x+1}{2}\ dx$

   $=\frac{4}{3}x^3+4x\sin x+4\cos x+\frac{1}{4}\sin 2x+\frac{x}{2}\ |_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$

   $=\frac{\sqrt{3}}{4}+\frac{\pi}{3}+\frac{8\pi^3}{81}$

   (e) $\int_0^{\pi}\ \frac{1}{\cos x}dx$

   $=\int_0^{\pi}\ \sec x\ dx$

   $=\lim_{a\to\frac{\pi}{2}^-}\ \int_0^a\ \sec x\ dx+\lim_{b\to\frac{\pi}{2}^+}\ \int_b^{\pi}\ \sec x\ dx$

   $=\lim_{a\to\frac{\pi}{2}^-}\ \int_0^a\ \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}\ dx+\lim_{b\to\frac{\pi}{2}^+}\ \int_b^{\pi}\ \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}\ dx$

$= \lim_{a\to\frac{\pi}{2}^-} \int_0^a \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}\ dx + \lim_{b\to\frac{\pi}{2}^+} \int_b^\pi \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}\ dx$

$= \lim_{a\to\frac{\pi}{2}^-} \ln|\sec x+\tan x|\ |_0^a + \lim_{b\to\frac{\pi}{2}^+} \ln|\sec x+\tan x|\ |_b^\pi$

$= \lim_{a\to\frac{\pi}{2}^-} \ln|\sec a+\tan a| - \lim_{b\to\frac{\pi}{2}^+} \ln|\sec b+\tan b|$

since $\lim_{a\to\frac{\pi}{2}^-} \ln|\sec a+\tan a|$ and $\lim_{b\to\frac{\pi}{2}^+} \ln|\sec b+\tan b|$ are divergent

hence $\int_0^\pi \frac{1}{\cos x}\ dx$ *is divergent*

8. (a) $\lim_{x\to1} \frac{x^2-1}{x-1} = \lim_{x\to1} \frac{(x+1)(x-1)}{x-1} = \lim_{x\to1} x+1 = 2$

(b) $\lim_{x\to\infty} \frac{x^3-x^2+4x-4}{x^3-6x^2+5x} = \lim_{x\to\infty} \frac{1-\frac{1}{x}+\frac{4}{x^2}-\frac{4}{x^3}}{1-\frac{6}{x}+\frac{5}{x^2}} = 1$

(c) $\lim_{x\to0} \frac{\sin(2x)}{5x}$ ($(\frac{0}{0})$ apply l'Hopital's rule)

$= \lim_{x\to0} \frac{2\cos(2x)}{5}$

$= \frac{2}{5}$

(d) $\lim_{x\to0^+} \frac{(\ln x)^4}{x} = \infty$