Calculus Examination2 Solution

TEST A

1. $f_x = e^{\sin x} y \cos x$

   $f_y = e^{\sin x}$

   $f_{xx} = e^{\sin x} y \cos^2 x - e^{\sin x} y \sin x$

   $f_{xy} = f_{yx} = e^{\sin x} \cos x$

   $f_{yy} = 0$

2. $f(x) = \frac{\sin x}{1+\cos^2 x}$

   $f'(x) = \frac{\cos x}{1+\cos^2 x} + \frac{2\cos x \sin^2 x}{(1+\cos^2 x)^2} = \frac{\cos x(1+\cos^2 x+2\sin^2 x)}{(1+\cos^2 x)^2} = \frac{\cos x(3-\cos^2 x)}{(1+\cos^2 x)^2}$

   $f'(x) = 0 \Rightarrow \cos x = 0 \Rightarrow x = \frac{k\pi}{2} \quad , k = \pm 1, \pm 2, \pm 3, \ldots$

   $f(\frac{\pi}{2} + 2k\pi) = 1$ is the relative maximum

   $f(-\frac{\pi}{2} + 2k\pi) = -1$ is the relative minimum

3. Let f(x) $= e^{2x} \tan(\frac{x}{2})$

   f'(x) $= 2e^{2x} \tan(\frac{x}{2}) + \frac{1}{2} e^{2x} \sec^2(\frac{x}{2})$

   f'(0) $= \frac{1}{2}$

   $\Rightarrow y = \frac{1}{2}x$ is the tangent line at point (0,0)

4. $\int_0^4 \int_0^4 (xy)^2 \, dx \, dy = \int_0^4 \frac{x^3}{3} y^2 \mid_0^4 dy = \frac{64}{3} \cdot \frac{y^3}{3} \mid_0^4 = \frac{4096}{9}$

5. (a) $a = \frac{4\times(100\times35+150\times44+200\times54+250\times66)-(100+150+200+250)\times(35+44+54+66)}{(100^2+150^2+200^2+250^2)-(100+150+200+250)^2} = 0.206$

   $b = \frac{1}{4} \cdot (35 + 44 + 54 + 66) - a(100 + 150 + 200 + 250) = 13.7$

   hence the least square regression line is $y = 0.206x + 13.7$

   (b) $0.206 \times 160 + 13.7 = 46.66$

6. (a) $\int_0^{12} \ 15 + 6\sin\frac{\pi(t-8)}{6} \ dt = 15t - \frac{36}{\pi}\cos\frac{\pi(t-8)}{6} \ |_0^{12} = 180 - \frac{36}{\pi}(-\frac{1}{2}) + \frac{36}{\pi}(-\frac{1}{2}) = 180$

(b) $\frac{1}{12-6}\int_6^{12} \ 15 + 6\sin\frac{\pi(t-8)}{6} \ dt$

$= \frac{1}{6}(15t - \frac{36}{\pi}\cos\frac{\pi(t-8)}{6}) \ |_6^{12}$

$= 15 - \frac{6}{\pi}(-\frac{1}{2}) + \frac{6}{\pi}(\frac{1}{2})$

$= 15 + \frac{6}{\pi}$

7. (a) $\int \ x\cos x \ dx = \int \ x \ d\sin x = x\sin x - \int \ \sin x \ dx = x\sin x + \cos x + c$

(b) $\int_0^3 \ \int_0^{x^2} \ \sqrt{1+x} \ dy \ dx$

$= \int_0^3 \ y\sqrt{1+x} \ |_0^{x^2} \ dx$

$= \int_0^3 \ x^2\sqrt{1+x} \ dx$

$= \int_0^3 \ (x+1-1)^2\sqrt{x+1} \ d(x+1)$

$= \int_0^3 \ (x+1)^{\frac{5}{2}} - 2(x+1)^{\frac{3}{2}} + (x+1)^{\frac{1}{2}} \ d(x+1)$

$= \frac{2}{7}(x+1)^{\frac{7}{2}} - \frac{4}{5}(x+1)^{\frac{5}{2}} + \frac{2}{3}(x+1)^{\frac{3}{2}} \ |_0^3$

$= \frac{1696}{105}$

(c) $\int \ \frac{\sin^2(\frac{x}{2})}{x-\sin x} \ dx = \frac{1}{2}\int \ \frac{1-\cos x}{x-\sin x} \ dx = \frac{1}{2}\ln|x-\sin x| + c$

(d) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \ (2x+\cos x)^2 \ dx$

$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \ 4x^2 + 4x\cos x + \cos^2 x \ dx$

$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \ 4x^2 + 4x\cos x + \frac{\cos 2x+1}{2} \ dx$

$= \frac{4}{3}x^3 + 4x\sin x + 4\cos x + \frac{1}{4}\sin 2x + \frac{x}{2} \ |_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$

$= \frac{1}{2} + \frac{\pi}{4} + \frac{\pi^3}{24}$

(e) $\int_0^{\pi} \ \frac{1}{\cos x}dx$

$= \int_0^{\pi} \ \sec x \ dx$

$= \lim_{a\to\frac{\pi}{2}^-} \ \int_0^a \ \sec x \ dx + \lim_{b\to\frac{\pi}{2}^+} \ \int_b^{\pi} \ \sec x \ dx$

$= \lim_{a\to\frac{\pi}{2}^-} \ \int_0^a \ \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x} \ dx + \lim_{b\to\frac{\pi}{2}^+} \ \int_b^{\pi} \ \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x} \ dx$

$= \lim_{a\to\frac{\pi}{2}^-} \int_0^a \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}\, dx + \lim_{b\to\frac{\pi}{2}^+} \int_b^\pi \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}\, dx$

$= \lim_{a\to\frac{\pi}{2}^-} \ln|\sec x + \tan x|\ |_0^a + \lim_{b\to\frac{\pi}{2}^+} \ln|\sec x + \tan x|\ |_b^\pi$

$= \lim_{a\to\frac{\pi}{2}^-} \ln|\sec a + \tan a| - \lim_{b\to\frac{\pi}{2}^+} \ln|\sec b + \tan b|$

since $\lim_{a\to\frac{\pi}{2}^-} \ln|\sec a + \tan a|$ and $\lim_{b\to\frac{\pi}{2}^+} \ln|\sec b + \tan b|$ are divergent

hence $\int_0^\pi \frac{1}{\cos x}\, dx$ *is divergent*

8. (a) $\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{x-1} = \lim_{x\to 1} x+1 = 2$

   (b) $\lim_{x\to\infty} \frac{x^3-x^2+4x-4}{x^3-6x^2+5x} = \lim_{x\to\infty} \frac{1-\frac{1}{x}+\frac{4}{x^2}-\frac{4}{x^3}}{1-\frac{6}{x}+\frac{5}{x^2}} = 1$

   (c) $\lim_{x\to 0} \frac{\sin 2x}{5x}$ ($(\frac{0}{0})$ apply l'Hopital's rule)

   $= \lim_{x\to 0} \frac{2\cos 2x}{5}$

   $= \frac{2}{5}$

   (d) $\lim_{x\to 0^+} \frac{(\ln x)^4}{x} = \infty$