Calculus Midterm #1 (Form D)

(1)

$$(f(x))^2 = \left(\frac{1}{x^p}\right)^2 = x^{-2p}.$$

(i) $p = \frac{1}{2}$,

$$\int_0^1 \pi(f(x))^2 dx = \lim_{b\to o^+}\int_b^1 \pi\frac{1}{x}dx = \pi\lim_{b\to 0^+}\Big[\ln x\Big]_b^1 = \pi\lim_{b\to 0^+}(\ln 1 - \ln b) = \infty.$$

(ii) $p \neq \frac{1}{2}$,

$$\begin{aligned}\int_0^1 \pi(f(x))^2 dx &= \lim_{b\to 0^+}\int_b^1 \pi x^{-2p}dx = \pi\lim_{b\to 0^+}\Big[\frac{x^{-2p+1}}{-2p+1}\Big]_b^1 \\ &= \frac{\pi}{1-2p}\lim_{b\to 0^+}(1-b^{1-2p}) \\ &= \begin{cases} \frac{\pi}{1-2p}(1-0) = \frac{\pi}{1-2p}, & \text{if } 1-2p>0; \\ \frac{\pi}{1-2p}(1-\infty) = \infty, & \text{if } 1-2p<0.\end{cases}\end{aligned}$$

Therefore, if $0 < p < \frac{1}{2}$, the solid has a finite volume $\frac{\pi}{1-2p.}$ ∎

(2)

(i) Begin by factoring the denominator $x(x+1)^2$. Then, write the partial fraction decomposition as

$$\frac{3x+1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

To solve this equation for A, B, and C, multiply each side of the equation by the least common denominator $x(x+1)^2$.

$$\begin{aligned}3x+1 &= A(x+1)^2 + Bx(x+1) + Cx \\ &= (Ax^2 + 2Ax + A) + (Bx^2 + Bx) + Cx \\ &= (A+B)x^2 + (2A+B+C)x + A.\end{aligned}$$

Hence, $A + B = 0$, $2A + B + C = 3$, and $1 = A$, which has the solution $A = 1$, $B = -1$, and $C = 2$. Therefore,

$$\begin{aligned}\int \frac{3x+1}{x(x+1)^2}dx &= \int \frac{1}{x} + \frac{-1}{x+1} + \frac{2}{(x+1)^2}dx \\ &= \int \frac{1}{x}\,dx + \int \frac{-1}{x+1}\,dx + \int \frac{2}{(x+1)^2}\,dx \\ &= \ln|x| - \ln|x+1| + 2(-1)\frac{1}{x+1} + C \\ &= \ln|x| - \ln|x+1| - \frac{2}{x+1} + C.\end{aligned}$$

∎

(ii) Consider the substitution $u = \frac{1}{x}$, which produces $du = -\frac{1}{x^2}dx$.

$$\int \frac{e^{1/x}}{x^2}\,dx = \int e^u(-1)\,du = -\int e^u\,du = -e^u + C = -e^{1/x} + C.$$

∎

(iii) Use integration by parts and let $dv = xdx$.

$$dv = xdx \quad \Rightarrow \quad v = \frac{x^2}{2}$$
$$u = (\ln x)^2 \quad \Rightarrow \quad du = 2(\ln x)(\frac{1}{x})dx$$

This implies that

$$\int x(\ln x)^2\, dx = \frac{x^2}{2}(\ln x)^2 - \int x \ln x\, dx.$$

To evaluate the integral on the right, apply integration by parts once again.

$$dv = xdx \quad \Rightarrow \quad v = \frac{x^2}{2}$$
$$u = \ln x \quad \Rightarrow \quad du = \frac{1}{x}dx$$

which gives

$$\begin{aligned}\int x(\ln x)^2\, dx &= \frac{x^2}{2}(\ln x)^2 - \int x \ln x\, dx \\ &= \frac{x^2}{2}(\ln x)^2 - \Big[\frac{x^2}{2}\ln x - \int \frac{x}{2}\, dx\Big] \\ &= \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4} + C.\end{aligned}$$

∎

(iv) Consider the substitution $u = 3x + 1$, which produces $du = 3dx$ and $x = \frac{u-1}{3}$. The lower and upper limits are changed to $u = 4$ and $u = 10$, respectively.

$$\begin{aligned}\int_1^3 \frac{x}{\sqrt{3x+1}}\, dx &= \int_4^{10} \frac{1}{\sqrt{u}}\frac{u-1}{3}\frac{1}{3}\, du \\ &= \frac{1}{9}\int_4^{10} (u-1)u^{-\frac{1}{2}}\, du \\ &= \frac{1}{9}\int_4^{10} u^{\frac{1}{2}} - u^{-\frac{1}{2}}\, du \\ &= \frac{1}{9}\Big[\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\Big]_4^{10} \\ &\approx \frac{1}{9}(14.7573 - 1.3333) \\ &= 1.4916.\end{aligned}$$

∎

(3) When $n = 6$, the width of each subinterval is $(1-(-1))/6 = \frac{1}{3}$ and the endpoints of the subintervals are

$$x_0 = -1,\;\; x_1 = -\frac{2}{3},\;\; x_2 = -\frac{1}{3},\;\; x_3 = 0,\;\; x_4 = \frac{1}{3},\;\; x_5 = \frac{2}{3},\;\; x_6 = 1.$$

So, by the Simpson Rule

$$\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx = \frac{1}{\sqrt{2\pi}}\Big(\frac{2}{18}\Big)\Big[e^{-\frac{(-1)^2}{2}} + 4e^{-\frac{(-\frac{2}{3})^2}{2}} + 2e^{-\frac{(-\frac{1}{3})^2}{2}} + 4e^{-\frac{(0)^2}{2}} + 2e^{-\frac{(\frac{1}{3})^2}{2}} + 4e^{-\frac{(\frac{2}{3})^2}{2}} + e^{-\frac{(1)^2}{2}}\Big] \approx 0.683.$$

∎

(4)

$$f(x,y) = e^{-2y/x}.$$

Begin by finding the first partial derivatives. Holding as a y constant, we obtain

$$f_x(x,y) = e^{-2y/x}\frac{\partial}{\partial x}\Big[\frac{-2y}{x}\Big] = e^{-2y/x}(-2y)(-\frac{1}{x^2}) = \frac{2y}{x^2}\, e^{-2y/x}.$$

Holding as a x constant, we obtain

$$f_y(x,y) = e^{-2y/x}\frac{\partial}{\partial y}\Big[\frac{-2y}{x}\Big] = e^{-2y/x}\Big(\frac{-2}{x}\Big) = \frac{-2}{x}e^{-2y/x}.$$

Then, differentiating f_x and f_y with respect to x and y to obtain the second partials as follows.

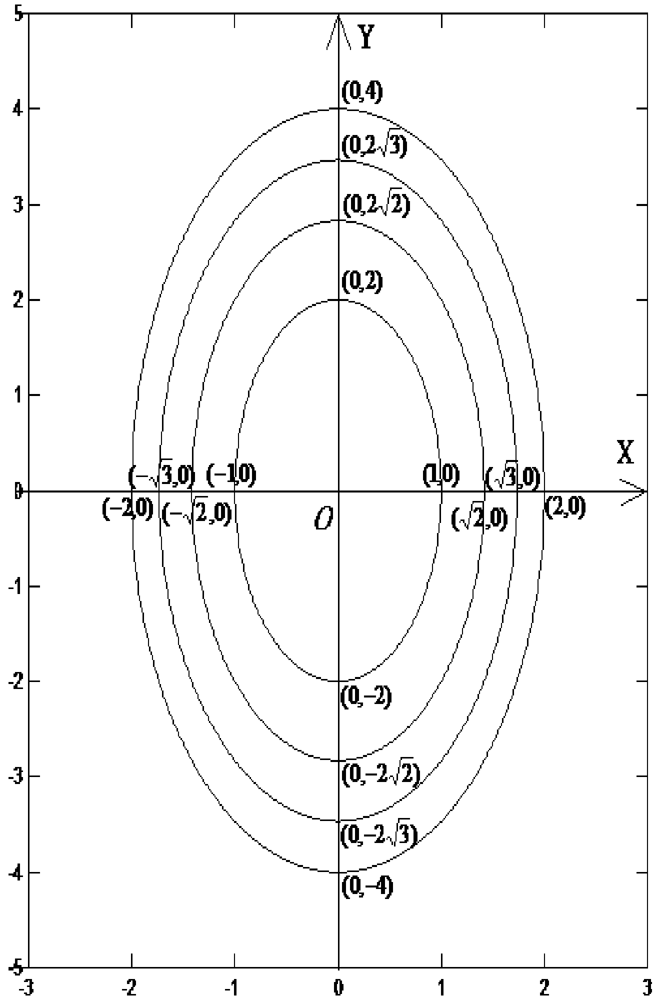
$$\begin{aligned}
f_{xx}(x,y) &= \Big[\frac{\partial}{\partial x}\Big(\frac{2y}{x^2}\Big)\Big]e^{-2y/x} + \frac{2y}{x^2}\Big[\frac{\partial}{\partial x}e^{-2y/x}\Big] \\
&= (-4yx^{-3})e^{-2y/x} + \frac{2y}{x^2}e^{-2y/x}(-2y)(-\frac{1}{x^2}) \\
&= e^{-2y/x}\Big(\frac{-4y}{x^3} + \frac{4y^2}{x^4}\Big) \\
&= -\frac{4y}{x^3}e^{-2y/x} + \frac{4y^2}{x^4}e^{-2y/x}, \\
f_{xy}(x,y) &= \Big[\frac{\partial}{\partial y}\Big(\frac{2y}{x^2}\Big)\Big]e^{-2y/x} + \frac{2y}{x^2}\Big[\frac{\partial}{\partial y}e^{-2y/x}\Big] \\
&= \frac{2}{x^2}e^{-2y/x} + \frac{2y}{x^2}e^{-2y/x}(-\frac{2}{x}) \\
&= \frac{2e^{-2y/x}}{x^2} - \frac{4y}{x^3}e^{-2y/x}, \\
f_{yy}(x,y) &= -\frac{2}{x}e^{-2y/x}\Big(-\frac{2}{x}\Big) = \frac{4}{x^2}e^{-2y/x}, \\
f_{yx}(x,y) &= \Big[\frac{\partial}{\partial x}\Big(\frac{-2}{x}\Big)\Big]e^{-2y/x} + \frac{-2}{x}\Big[\frac{\partial}{\partial x}e^{-2y/x}\Big] \\
&= (-2)(-1)x^{-2}e^{-2y/x} + \frac{-2}{x}e^{-2y/x}(-2y)(-1)x^{-2} \\
&= \frac{2}{x^2}e^{-2y/x} - \frac{4y}{x^3}e^{-2y/x}.
\end{aligned}$$

∎

(5)

(i)

$$f(x,y) = x^2 + \frac{y^2}{4} - 1 = 0 \;\Rightarrow\; x^2 + \frac{y^2}{4} = 1 \;\Rightarrow\; x^2 + \Big(\frac{y}{2}\Big)^2 = 1,$$
$$f(x,y) = x^2 + \frac{y^2}{4} - 1 = 1 \;\Rightarrow\; x^2 + \frac{y^2}{4} = 2 \;\Rightarrow\; \Big(\frac{x}{\sqrt{2}}\Big)^2 + \Big(\frac{y}{2\sqrt{2}}\Big)^2 = 1,$$
$$f(x,y) = x^2 + \frac{y^2}{4} - 1 = 2 \;\Rightarrow\; x^2 + \frac{y^2}{4} = 3 \;\Rightarrow\; \Big(\frac{x}{\sqrt{3}}\Big)^2 + \Big(\frac{y}{2\sqrt{3}}\Big)^2 = 1,$$
$$f(x,y) = x^2 + \frac{y^2}{4} - 1 = 3 \;\Rightarrow\; x^2 + \frac{y^2}{4} = 4 \;\Rightarrow\; \Big(\frac{x}{2}\Big)^2 + \Big(\frac{y}{4}\Big)^2 = 1,$$



(ii)

$$f_x(x,y) = 2x,$$

$f_x(1,0) = 2,$ $\qquad f_x(-1,0) = -2,$

$f_x(\sqrt{2},0) = 2\sqrt{2},$ $\qquad f_x(-\sqrt{2},0) = -2\sqrt{2},$

$f_x(\sqrt{3},0) = 2\sqrt{3},$ $\qquad f_x(-\sqrt{3},0) = -2\sqrt{3},$

$f_x(2,0) = 4,$ $\qquad f_x(-2,0) = -4,$

(iii) Using (i), the height gap is 1 in z-axis, it means the gap is equal. But the contour map presents more and more tight squeeze in x-axis, $1 > \sqrt{2}-1 > \sqrt{3}-\sqrt{2} > 2-\sqrt{3}$, so the surface becomes steeper in the direction of the

x-axis. Using (ii), the slope is more and more bigger, it means that the surface becomes steeper in the direction of the x-axis. ∎

(6)

(i) According to the supply and demand principle, when the unit price for p_1 is increased, the numbers of units sold for x_1 is decreased. But, $\frac{\partial g}{\partial p_1} > 0$, implies that the unit price for p_1 increases, the numbers of units sold for x_2 also increases. Therefore, they are substitute.

(ii)

$$\frac{\partial x_2}{\partial p_1} = \frac{750}{p_2}\frac{\partial p_1^{-\frac{1}{2}}}{\partial p_1} = \frac{750}{p_2}\Big(-\frac{1}{2}\Big)p_1^{-\frac{3}{2}} = -\frac{375}{p_2\sqrt{p_1^3}} < 0.$$

Then, they are complementary. ∎