Calculus Midterm #1 (Form C)

(1)

$$(f(x))^2 = \Big(\frac{1}{x^p}\Big)^2 = x^{-2p}.$$

(i) $p = \frac{1}{2}$,

$$\int_1^\infty \pi (f(x))^2 dx = \lim_{b\to\infty} \int_1^b \pi \frac{1}{x} dx = \pi \lim_{b\to\infty} \Big[\ln x\Big]_1^b = \pi \lim_{b\to\infty} (\ln b - \ln 1) = \infty.$$

(ii) $p \neq \frac{1}{2}$,

$$\begin{aligned} \int_1^\infty \pi (f(x))^2 dx &= \lim_{b\to\infty} \int_1^b \pi x^{-2p} dx = \pi \lim_{b\to\infty} \Big[\frac{x^{-2p+1}}{-2p+1}\Big]_1^b \\ &= \frac{\pi}{1-2p} \lim_{b\to\infty} (b^{1-2p} - 1) \\ &= \begin{cases} \infty, & \text{if } 1-2p > 0; \\ \frac{\pi}{1-2p}(0-1) = \frac{\pi}{2p-1}, & \text{if } 1-2p < 0. \end{cases} \end{aligned}$$

Therefore, if $p > \frac{1}{2}$, the solid has a finite volume $\frac{\pi}{2p-1.}$ ∎

(2)

(i) Begin by factoring the denominator $x(x+1)^2$. Then, write the partial fraction decomposition as

$$\frac{x-1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}.$$

To solve this equation for A, B, and C, multiply each side of the equation by the least common denominator $x^2(x+1)$.

$$\begin{aligned} x - 1 &= Ax(x+1) + B(x+1) + Cx^2 \\ &= (Ax^2 + Ax) + (Bx + B) + Cx^2 \\ &= (A + C)x^2 + (A + B)x + B. \end{aligned}$$

Hence, $A + C = 0$, $A + B = 1$, and $B = -1$, which has the solution $A = 2$, $B = -1$, and $C = -2$. Therefore,

$$\begin{aligned} \int \frac{3x+1}{x(x+1)^2} dx &= \int \frac{2}{x} + \frac{-1}{x^2} + \frac{-2}{x+1} dx \\ &= \int \frac{2}{x}\, dx + \int \frac{-1}{x^2}\, dx + \int \frac{-2}{x+1}\, dx \\ &= 2\ln|x| - (-1)x^{-1} - 2\ln|x+1| + C \\ &= 2\ln|x| + \frac{1}{x} - 2\ln|x+1| + C. \end{aligned}$$

∎

(ii) Consider the substitution $u = \frac{1}{x}$, which produces $du = -\frac{1}{x^2}dx$.

$$\int \frac{e^{1/x}}{x^2}\, dx = \int e^u(-1)\, du = -\int e^u\, du = -e^u + C = -e^{1/x} + C.$$

∎

(iii) Use integration by parts and let $dv = xdx$.

$$dv = xdx \quad \Rightarrow \quad v = \frac{x^2}{2}$$
$$u = (\ln x)^2 \quad \Rightarrow \quad du = 2(\ln x)(\frac{1}{x})dx$$

This implies that

$$\int x(\ln x)^2\, dx = \frac{x^2}{2}(\ln x)^2 - \int x \ln x\, dx.$$

To evaluate the integral on the right, apply integration by parts once again.

$$dv = xdx \quad \Rightarrow \quad v = \frac{x^2}{2}$$
$$u = \ln x \quad \Rightarrow \quad du = \frac{1}{x}dx$$

which gives

$$\begin{aligned}\int x(\ln x)^2\, dx &= \frac{x^2}{2}(\ln x)^2 - \int x \ln x\, dx \\ &= \frac{x^2}{2}(\ln x)^2 - \big[\frac{x^2}{2}\ln x - \int \frac{x}{2}\, dx\big] \\ &= \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4} + C.\end{aligned}$$

∎

(iv) Consider the substitution $u = 2x + 1$, which produces $du = 2dx$ and $x = \frac{u-1}{2}$. The lower and upper limits are changed to $u = 3$ and $u = 7$, respectively.

$$\begin{aligned}\int_1^3 \frac{x}{\sqrt{2x+1}}\, dx &= \int_3^7 \frac{1}{\sqrt{u}} \frac{u-1}{2}\frac{1}{2}\, du \\ &= \frac{1}{4}\int_3^7 (u-1)u^{-\frac{1}{2}}\, du \\ &= \frac{1}{4}\int_3^7 u^{\frac{1}{2}} - u^{-\frac{1}{2}}\, du \\ &= \frac{1}{4}\Big[\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\Big]_3^7 \\ &\approx \frac{1}{4}(7.0553 - 0) \\ &= 1.7638.\end{aligned}$$

∎

(3) When $n = 6$, the width of each subinterval is $(1-(-1))/6 = \frac{1}{3}$ and the endpoints of the subintervals are

$$x_0 = -1,\ \ x_1 = -\frac{2}{3},\ \ x_2 = -\frac{1}{3},\ \ x_3 = 0,\ \ x_4 = \frac{1}{3},\ \ x_5 = \frac{2}{3},\ \ x_6 = 1.$$

So, by the Trapezoidal Rule

$$\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx = \frac{1}{\sqrt{2\pi}}\Big(\frac{2}{12}\Big)\Big[e^{-\frac{(-1)^2}{2}} + 2e^{-\frac{(-\frac{2}{3})^2}{2}} + 2e^{-\frac{(-\frac{1}{3})^2}{2}} + 2e^{-\frac{(0)^2}{2}} + 2e^{-\frac{(\frac{1}{3})^2}{2}} + 2e^{-\frac{(\frac{2}{3})^2}{2}} + e^{-\frac{(1)^2}{2}}\Big] \approx 0.678.$$

∎

(4)

$$f(x,y) = e^{-2x/y}.$$

Begin by finding the first partial derivatives. Holding as a y constant, we obtain

$$f_x(x,y) = e^{-2x/y}\frac{\partial}{\partial x}\Big[\frac{-2x}{y}\Big] = e^{-2x/y}\Big(\frac{-2}{y}\Big) = \frac{-2}{y}e^{-2x/y}.$$

Holding as a x constant, we obtain

$$f_y(x,y) = e^{-2x/y}\frac{\partial}{\partial y}\Big[\frac{-2x}{y}\Big] = e^{-2x/y}(-2x)(-\frac{1}{y^2}) = \frac{2x}{y^2}\ e^{-2x/y}.$$

Then, differentiating f_x and f_y with respect to x and y to obtain the second partials as follows.

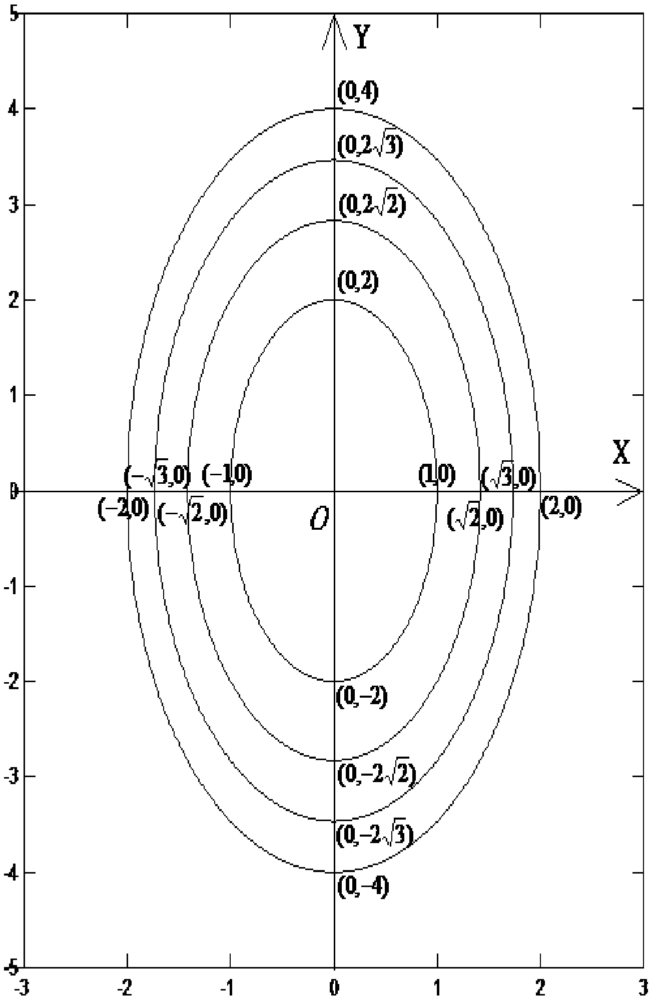
$$\begin{aligned}
f_{xx}(x,y) &= -\frac{2}{y}e^{-2x/y}\Big(-\frac{2}{y}\Big) = \frac{4}{y^2}e^{-2x/y},\\
f_{xy}(x,y) &= \Big[\frac{\partial}{\partial y}\Big(\frac{-2}{y}\Big)\Big]e^{-2x/y} + \frac{-2}{y}\Big[\frac{\partial}{\partial y}e^{-2x/y}\Big]\\
&= (-2)(-1)y^{-2}e^{-2x/y} + \frac{-2}{y}e^{-2x/y}(-2x)(-1)x^{-2}\\
&= \frac{2}{y^2}e^{-2x/y} - \frac{4x}{y^3}e^{-2x/y},\\
f_{yy}(x,y) &= \Big[\frac{\partial}{\partial y}\Big(\frac{2x}{y^2}\Big)\Big]e^{-2x/y} + \frac{2x}{y^2}\Big[\frac{\partial}{\partial y}e^{-2x/y}\Big]\\
&= (-4xy^{-3})e^{-2x/y} + \frac{2x}{y^2}e^{-2x/y}(-2x)(-\frac{1}{y^2})\\
&= e^{-2x/y}\Big(\frac{-4x}{y^3} + \frac{4x^2}{y^4}\Big)\\
&= -\frac{4x}{y^3}e^{-2x/y} + \frac{4x^2}{y^4}e^{-2x/y},\\
f_{yx}(x,y) &= \Big[\frac{\partial}{\partial x}\Big(\frac{2x}{y^2}\Big)\Big]e^{-2x/y} + \frac{2x}{y^2}\Big[\frac{\partial}{\partial x}e^{-2x/y}\Big]\\
&= \frac{2}{y^2}e^{-2x/y} + \frac{2x}{y^2}e^{-2x/y}(-\frac{2}{y})\\
&= \frac{2e^{-2x/y}}{y^2} - \frac{4x}{y^3}e^{-2x/y}.
\end{aligned}$$

∎

(5)

(i)

$$f(x,y) = x^2 + \frac{y^2}{4} - 1 = 0 \;\Rightarrow\; x^2 + \frac{y^2}{4} = 1 \;\Rightarrow\; x^2 + \Big(\frac{y}{2}\Big)^2 = 1,$$
$$f(x,y) = x^2 + \frac{y^2}{4} - 1 = 1 \;\Rightarrow\; x^2 + \frac{y^2}{4} = 2 \;\Rightarrow\; \Big(\frac{x}{\sqrt{2}}\Big)^2 + \Big(\frac{y}{2\sqrt{2}}\Big)^2 = 1,$$
$$f(x,y) = x^2 + \frac{y^2}{4} - 1 = 2 \;\Rightarrow\; x^2 + \frac{y^2}{4} = 3 \;\Rightarrow\; \Big(\frac{x}{\sqrt{3}}\Big)^2 + \Big(\frac{y}{2\sqrt{3}}\Big)^2 = 1,$$
$$f(x,y) = x^2 + \frac{y^2}{4} - 1 = 3 \;\Rightarrow\; x^2 + \frac{y^2}{4} = 4 \;\Rightarrow\; \Big(\frac{x}{2}\Big)^2 + \Big(\frac{y}{4}\Big)^2 = 1,$$



(ii)

$$f_y(x,y) = \frac{1}{4}(2y) = \frac{y}{2},$$

$f_y(0,2) = 1,$ $\quad f_y(0,-2) = -1,$

$f_y(0,2\sqrt{2}) = \sqrt{2},$ $\quad f_y(0,-2\sqrt{2}) = -\sqrt{2},$

$f_y(0,2\sqrt{3}) = \sqrt{3},$ $\quad f_y(0,-2\sqrt{3}) = -\sqrt{3},$

$f_y(0,4) = 2,$ $\quad f_y(0,-4) = -2,$

(iii) Using (i), the height gap is 1 in z-axis, it means the gap is equal. But the contour map presents more and more tight squeeze in y-axis, $2 > 2\sqrt{2}-2 > 2\sqrt{3} - 2\sqrt{2} > 4 - 2\sqrt{3}$, so the surface becomes steeper in the direction of the y-axis. Using (ii), the slope is more and more bigger, it means that the surface becomes steeper in the direction of the y-axis. ∎

(6)

(i) According to the supply and demand principle, when the unit price for p_1 is increased, the numbers of units sold for x_1 is decreased. But, $\frac{\partial g}{\partial p_1} > 0$, implies that the unit price for p_1 increases, the numbers of units sold for x_2 also increases. Therefore, they are substitute.

(ii)

$$\frac{\partial x_2}{\partial p_1} = \frac{750}{p_2}\frac{\partial p_1^{-\frac{1}{2}}}{\partial p_1} = \frac{750}{p_2}\Big(-\frac{1}{2}\Big)p_1^{-\frac{3}{2}} = -\frac{375}{p_2\sqrt{p_1^3}} < 0.$$

Then, they are complementary. ∎