Calculus Midterm #1 (Form A)

(1)

$$(f(x))^2 = \Big(\frac{1}{x^p}\Big)^2 = x^{-2p}.$$

(i) $p = \frac{1}{2}$,

$$\int_1^\infty \pi(f(x))^2 dx = \lim_{b\to\infty}\int_1^b \pi\frac{1}{x}dx = \pi\lim_{b\to\infty}\Big[\ln x\Big]_1^b = \pi\lim_{b\to\infty}(\ln b - \ln 1) = \infty.$$

(ii) $p \neq \frac{1}{2}$,

$$\begin{aligned}\int_1^\infty \pi(f(x))^2 dx &= \lim_{b\to\infty}\int_1^b \pi x^{-2p}dx = \pi\lim_{b\to\infty}\Big[\frac{x^{-2p+1}}{-2p+1}\Big]_1^b \\ &= \frac{\pi}{1-2p}\lim_{b\to\infty}(b^{1-2p}-1) \\ &= \begin{cases}\infty, & \text{if } 1-2p>0; \\ \frac{\pi}{1-2p}(0-1) = \frac{\pi}{2p-1}, & \text{if } 1-2p<0.\end{cases}\end{aligned}$$

Therefore, if $p > \frac{1}{2}$, the solid has a finite volume $\frac{\pi}{2p-1.}$ ∎

(2)

(i) Begin by factoring the denominator $x(x+1)^2$. Then, write the partial fraction decomposition as

$$\frac{3x+1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

To solve this equation for $A$, $B$, and $C$, multiply each side of the equation by the least common denominator $x(x+1)^2$.

$$\begin{aligned}3x+1 &= A(x+1)^2 + Bx(x+1) + Cx \\ &= (Ax^2+2Ax+A) + (Bx^2+Bx) + Cx \\ &= (A+B)x^2 + (2A+B+C)x + A.\end{aligned}$$

Hence, $A+B=0$, $2A+B+C=3$, and $1=A$, which has the solution $A=1$, $B=-1$, and $C=2$. Therefore,

$$\begin{aligned}\int\frac{3x+1}{x(x+1)^2}dx &= \int\frac{1}{x}+\frac{-1}{x+1}+\frac{2}{(x+1)^2}dx \\ &= \int\frac{1}{x}\,dx + \int\frac{-1}{x+1}\,dx + \int\frac{2}{(x+1)^2}\,dx \\ &= \ln|x| - \ln|x+1| + 2(-1)\frac{1}{x+1} + C \\ &= \ln|x| - \ln|x+1| - \frac{2}{x+1} + C.\end{aligned}$$

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(ii) Consider the substitution $u = \frac{1}{x}$, which produces $du = -\frac{1}{x^2}dx$.

$$\int\frac{e^{1/x}}{x^2}\,dx = \int e^u(-1)\,du = -\int e^u\,du = -e^u + C = -e^{1/x} + C.$$

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(iii) Use integration by parts and $dv = xdx$.

$$dv = xdx \quad \Rightarrow \quad v = \frac{x^2}{2}$$
$$u = (\ln x)^2 \quad \Rightarrow \quad du = 2(\ln x)(\frac{1}{x})dx$$

This implies that

$$\int x(\ln x)^2 \, dx = \frac{x^2}{2}(\ln x)^2 - \int x \ln x \, dx.$$

To evaluate the integral on the right, apply integration by parts once again.

$$dv = xdx \quad \Rightarrow \quad v = \frac{x^2}{2}$$
$$u = \ln x \quad \Rightarrow \quad du = \frac{1}{x}dx$$

which gives

$$\begin{aligned} \int x(\ln x)^2 \, dx &= \frac{x^2}{2}(\ln x)^2 - \int x \ln x \, dx \\ &= \frac{x^2}{2}(\ln x)^2 - \Big[\frac{x^2}{2}\ln x - \int \frac{x}{2} \, dx\Big] \\ &= \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4} + C. \end{aligned}$$

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(iv) Consider the substitution $u = 3x + 1$, which produces $du = 3dx$ and $x = \frac{u-1}{3}$. The lower and upper limits are changed to $u = 4$ and $u = 10$, respectively.

$$\begin{aligned} \int_1^3 \frac{x}{\sqrt{3x+1}} \, dx &= \int_4^{10} \frac{1}{\sqrt{u}} \frac{u-1}{3} \frac{1}{3} \, du \\ &= \frac{1}{9} \int_4^{10} (u-1)u^{-\frac{1}{2}} \, du \\ &= \frac{1}{9} \int_4^{10} u^{\frac{1}{2}} - u^{-\frac{1}{2}} \, du \\ &= \frac{1}{9} \Big[\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\Big]_4^{10} \\ &\approx \frac{1}{9}(14.7573 - 1.3333) \\ &= 1.4916. \end{aligned}$$

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(3) When $n = 6$, the width of each subinterval is $(1-(-1))/6 = \frac{1}{3}$ and the endpoints of the subintervals are

$$x_0 = -1, \;\; x_1 = -\frac{2}{3}, \;\; x_2 = -\frac{1}{3}, \;\; x_3 = 0, \;\; x_4 = \frac{1}{3}, \;\; x_5 = \frac{2}{3}, \;\; x_6 = 1.$$

So, by the Trapezoidal Rule

$$\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx = \frac{1}{\sqrt{2\pi}}\Big(\frac{2}{12}\Big)\Big[e^{-\frac{(-1)^2}{2}} + 2e^{-\frac{(-\frac{2}{3})^2}{2}} + 2e^{-\frac{(-\frac{1}{3})^2}{2}} + 2e^{-\frac{(0)^2}{2}} + 2e^{-\frac{(\frac{1}{3})^2}{2}} + 2e^{-\frac{(\frac{2}{3})^2}{2}} + e^{-\frac{(1)^2}{2}}\Big] \approx 0.678.$$

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(4)

$$f(x,y) = e^{-2y/x}.$$

Begin by finding the first partial derivatives. Holding $y$ as a constant, we obtain

$$f_x(x,y) = e^{-2y/x}\frac{\partial}{\partial x}\Big[\frac{-2y}{x}\Big] = e^{-2y/x}(-2y)(-\frac{1}{x^2}) = \frac{2y}{x^2}\, e^{-2y/x}.$$

Holding $x$ as a constant, we obtain

$$f_y(x,y) = e^{-2y/x}\frac{\partial}{\partial y}\Big[\frac{-2y}{x}\Big] = e^{-2y/x}\Big(\frac{-2}{x}\Big) = \frac{-2}{x}e^{-2y/x}.$$

Then, differentiating $f_x$ and $f_y$ with respect to $x$ and $y$ to obtain the second partials as follows.

$$\begin{aligned}
f_{xx}(x,y) &= \Big[\frac{\partial}{\partial x}\Big(\frac{2y}{x^2}\Big)\Big]e^{-2y/x} + \frac{2y}{x^2}\Big[\frac{\partial}{\partial x}e^{-2y/x}\Big] \\
&= (-4yx^{-3})e^{-2y/x} + \frac{2y}{x^2}e^{-2y/x}(-2y)(-\frac{1}{x^2}) \\
&= e^{-2y/x}\Big(\frac{-4y}{x^3} + \frac{4y^2}{x^4}\Big) \\
&= -\frac{4y}{x^3}e^{-2y/x} + \frac{4y^2}{x^4}e^{-2y/x}, \\
f_{xy}(x,y) &= \Big[\frac{\partial}{\partial y}\Big(\frac{2y}{x^2}\Big)\Big]e^{-2y/x} + \frac{2y}{x^2}\Big[\frac{\partial}{\partial y}e^{-2y/x}\Big] \\
&= \frac{2}{x^2}e^{-2y/x} + \frac{2y}{x^2}e^{-2y/x}(-\frac{2}{x}) \\
&= \frac{2e^{-2y/x}}{x^2} - \frac{4y}{x^3}e^{-2y/x}, \\
f_{yy}(x,y) &= -\frac{2}{x}e^{-2y/x}\Big(-\frac{2}{x}\Big) = \frac{4}{x^2}e^{-2y/x}, \\
f_{yx}(x,y) &= \Big[\frac{\partial}{\partial x}\Big(\frac{-2}{x}\Big)\Big]e^{-2y/x} + \frac{-2}{x}\Big[\frac{\partial}{\partial x}e^{-2y/x}\Big] \\
&= (-2)(-1)x^{-2}e^{-2y/x} + \frac{-2}{x}e^{-2y/x}(-2y)(-1)x^{-2} \\
&= \frac{2}{x^2}e^{-2y/x} - \frac{4y}{x^3}e^{-2y/x}.
\end{aligned}$$

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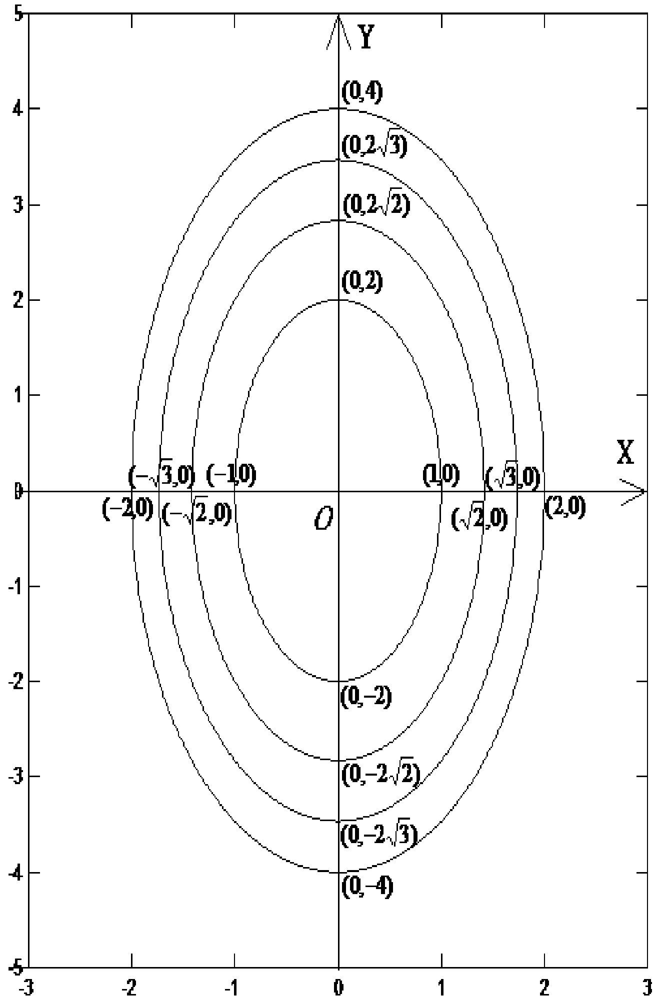
(5)

(i)

$$f(x,y) = x^2 + \frac{y^2}{4} - 1 = 0 \;\Rightarrow\; x^2 + \frac{y^2}{4} = 1 \;\Rightarrow\; x^2 + \left(\frac{y}{2}\right)^2 = 1,$$
$$f(x,y) = x^2 + \frac{y^2}{4} - 1 = 1 \;\Rightarrow\; x^2 + \frac{y^2}{4} = 2 \;\Rightarrow\; \left(\frac{x}{\sqrt{2}}\right)^2 + \left(\frac{y}{2\sqrt{2}}\right)^2 = 1,$$
$$f(x,y) = x^2 + \frac{y^2}{4} - 1 = 2 \;\Rightarrow\; x^2 + \frac{y^2}{4} = 3 \;\Rightarrow\; \left(\frac{x}{\sqrt{3}}\right)^2 + \left(\frac{y}{2\sqrt{3}}\right)^2 = 1,$$
$$f(x,y) = x^2 + \frac{y^2}{4} - 1 = 3 \;\Rightarrow\; x^2 + \frac{y^2}{4} = 4 \;\Rightarrow\; \left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2 = 1,$$




(ii)

$$f_y(x,y) = \frac{1}{4}(2y) = \frac{y}{2},$$

$f_y(0,2) = 1,$ $\quad f_y(0,-2) = -1,$

$f_y(0,2\sqrt{2}) = \sqrt{2},$ $\quad f_y(0,-2\sqrt{2}) = -\sqrt{2},$

$f_y(0,2\sqrt{3}) = \sqrt{3},$ $\quad f_y(0,-2\sqrt{3}) = -\sqrt{3},$

$f_y(0,4) = 2,$ $\quad f_y(0,-4) = -2,$

(iii) Using (i), the height gap is 1 in $z$-axis, it means the gap is equal. But the contour map presents more and more tight squeeze in $y$-axis, $2 > 2\sqrt{2}-2 > 2\sqrt{3}-2\sqrt{2} > 4-2\sqrt{3}$, so the surface becomes steeper in the direction of the $y$-axis. Using (ii), the slope is more and more bigger, it means that the surface becomes steeper in the direction of the $y$-axis. ∎

(6)

(i)

According to the supply and demand principle, when the unit price for $p_2$ is increased, the numbers of units sold for $x_2$ is decreased. But, $\frac{\partial f}{\partial p_2} > 0$, implies that the unit price for $p_2$ increases, the numbers of units sold for $x_1$ also increases. Therefore, they are substitute.

(ii)

$$\frac{\partial x_1}{\partial p_2} = \frac{1000}{\sqrt{p_1}} \frac{\partial p_2^{-\frac{1}{2}}}{\partial p_2} = \frac{1000}{\sqrt{p_1}} \Big(-\frac{1}{2}\Big) p_2^{-\frac{3}{2}} = -\frac{500}{\sqrt{p_1 p_2^3}} < 0.$$

Then, they are complementary. ∎