試題及答案卷

學系：____________ 學號：____________ 姓名：____________

注意：禁止使用計算器。

問題	1: 10分	2: 10分	3: 10分	4: 10分	5: 10分	總分: 100分
得分						
問題	6: 10分	7: 10分	8: 10分	9: 10分	10: 10分	
得分						

1. 給定函數

$$f(x) = \begin{cases} \frac{\sin(x^2)}{x^2}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

(a) f 是否在0點連續?爲什麼? 答案: Yes

(b) f 是否在0點可微分?爲什麼? 答案: Yes

解答:

(a) Yes.

$$\begin{aligned} \lim_{x\to 0} f(x) &= \lim_{x\to 0} \frac{\sin x^2}{x^2} \\ &= \lim_{x\to 0} \frac{x^2 + o(x^2)}{x^2} \\ &= 1 = f(0) \end{aligned}$$

(b) Yes.

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{\frac{\sin h^2}{h^2} - 1}{h} \\ &= \lim_{h\to 0} \frac{2h\cos h^2 - 2h}{3h^2} \\ &= \lim_{h\to 0} \frac{2\cos h^2 - 4h^2 \sin h^2 - 2}{6h} \\ &= \lim_{h\to 0} \frac{-2\sin h^2 - 4h^2 \sin h^2}{6h} = 0 \end{aligned}$$

□

2. 設$y = f(x)$是一個可微分函數且滿足下列式子:

$$y = x^{x+y}$$

求$y'(1)$。 答案: 2

解答: Note that if $x = 1$, then $y = 1$ since $y = 1^{1+y} = 1$. Taking derivative after taking ln on both sides yields $\ln y = (x+y)\ln x$, $\frac{y'}{y} = (1+y')\ln x + (x+y)\frac{1}{x}$. Hence $y' = \frac{\frac{xy+y^2}{x} + y\ln x}{1 - y\ln x}$ and $y'(1) = 2$. □

3. 證明對所有實數x, $f(x) = 3x^4 + 4x^3 + 1$恆大於等於0。 答案: $f(x) \geq f(-1) = 0$

解答: $f'(x) = 12x^3 + 12x^2 = 12x^2(x+1)$ and it attains its the minimal value 0 at $x = -1$. □

4. 設$g(x)$ 是一個定義在$x > -1$ 上的函數，且滿足下列式子:

$$g(3x^4 + 4x^3 + 1) = \ln(x+2)$$

求$g'(8)$ 之值。 答案: 1/72

解答: $(dg/dy)(12x^3 + 12x^2) = \frac{1}{x+2}, g'(8) = \left.\frac{\frac{1}{x+2}}{12x^3+12x^2}\right|_{x=1} = 1/72.$ □

5. 寫出所有連續函數f 滿足$(f(x))^2 = x^2$, $x \in \mathbb{R}$，並證明你已經寫出全部的這種連續函數。 答案: $\pm x$; $\pm|x|$

解答: There are 4 of them: $f_1(x) = x; f_2(x) = -x; f_3(x) = |x|; f_4(x) = -|x|$. $(f(x))^2 = x^2$, so $f(x) = \pm x$. If $f(x_1) > 0$ and $f(x_2) < 0$ for some $x_1 > 0, x_2 > 0$ then by intermediate value theorem for continuous function there exists a real number c between x_1 and x_2 (then $c > 0$) such that $f(c) = 0$, which contradicts to $(f(c))^2 = c^2$. Hence if $f(x_1) = x_1$ for some $x_1 > 0$ then $f(x) = x$ for all $x > 0$. □

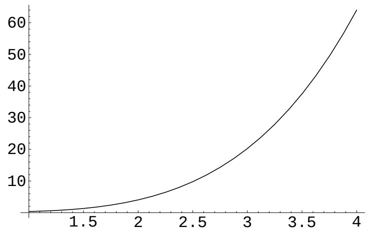
6. 試求$\int_1^3 |x^2 - 4|\,\mathrm{d}x$。 答案: 4

解答: 因爲

$$|x^2 - 4| = \begin{cases} 4 - x^2, & \text{如果} 1 \leq x \leq 2; \\ x^2 - 4, & \text{如果} 2 \leq x \leq 3; \end{cases}$$

因此 $\int_1^3 |x^2 - 4|\,\mathrm{d}x = \int_1^2 (4 - x^2)\,\mathrm{d}x + \int_2^3 (x^2 - 4)\,\mathrm{d}x = 4$。 □

7. 求曲線$y = \frac{x^4}{4} + \frac{1}{8x^2}$在$x = 1$和$x = 4$之間的長度。 答案: $\frac{8175}{128} = 63\frac{111}{128}$



解答: 令$f(x)=\frac{x^4}{4}+\frac{1}{8x^2}$, $L=\int_1^4\sqrt{1+[f'(x)]^2}\,\mathrm{d}x=\int_1^4\sqrt{1+(x^3-\frac{1}{4x^3})^2}\,\mathrm{d}x=\int_1^4 x^3+\frac{1}{4x^3}\,\mathrm{d}x=\frac{1}{4}x^4-\frac{1}{8x^2}\Big|_1^4=\frac{8175}{128}=63\frac{111}{128}$ □

8. 試求$\int_0^{\pi/2}\sin^2 x\cos^4 x\,\mathrm{d}x$。 答案: $\frac{\pi}{32}$

解答:

$$\begin{aligned}\int_0^{\pi/2}\sin^2 x\cos^4 x\,\mathrm{d}x &= \int_0^{\pi/2}(\sin x\cos x)^2\cos^2 x\,\mathrm{d}x\\ &= \int_0^{\pi/2}\left(\frac{1}{2}\sin 2x\right)^2\frac{1}{2}(1+\cos 2x)\,\mathrm{d}x\\ &= \frac{1}{8}\int_0^{\pi/2}\sin^2 2x\,\mathrm{d}x+\frac{1}{8}\int_0^{\pi/2}\sin^2 2x\cos 2x\,\mathrm{d}x\\ &= \frac{1}{16}\int_0^{\pi/2}(1-\cos 4x)\,\mathrm{d}x+\frac{1}{48}\sin^3 2x\Big|_0^{\pi/2}\\ &= \frac{1}{16}\left[x-\frac{\sin 4x}{4}\right]\Big|_0^{\pi/2}+0\\ &= \frac{\pi}{32}\end{aligned}$$ □

9. 計算 答案: $\frac{\pi}{6}$

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{4n^2-1^2}}+\frac{1}{\sqrt{4n^2-2^2}}+\cdots+\frac{1}{\sqrt{4n^2-n^2}}\right]$$

解答: 我們有

$$\begin{aligned}s_n &= \frac{1}{\sqrt{4n^2-1^2}}+\frac{1}{\sqrt{4n^2-2^2}}+\cdots+\frac{1}{\sqrt{4n^2-n^2}}\\ &= \frac{1}{n}\left[\frac{1}{\sqrt{4-(\frac{1}{n})^2}}+\frac{1}{\sqrt{4-(\frac{2}{n})^2}}+\cdots+\frac{1}{\sqrt{4-(\frac{n}{n})^2}}\right]\end{aligned}$$

因此s_n是函數$f:[0,1]\to\mathbb{R}, f(x)=\frac{1}{\sqrt{4-x^2}}$，對於$x_0=0<x_1=\frac{1}{n}<x_2=\frac{2}{n}<\cdots<x_n=\frac{n}{n}=1$分割，且中間點$\xi_i=\frac{i}{n}\in[x_i,x_{i+1}]$的黎曼和。所以其解爲

$$\lim_{n\to\infty}s_n=\int_0^1\frac{1}{\sqrt{4-x^2}}\,\mathrm{d}x=\arcsin\frac{x}{2}\Big|_0^1=\frac{\pi}{6}$$ □

10. 證明對任意的實數x，下列的級數和收斂並計算其和　　答案: $\frac{1}{2}(\cos x + \cosh x)$

$$1 + \frac{x^4}{4!} + \frac{x^8}{8!} + \frac{x^{12}}{12!} + \cdots$$

解答: 由以下泰勒展開式

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots$$
$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots$$

可以知道題目給定之級數和爲$\frac{1}{2}(\cos x + \cosh x)$的泰勒展開式。 □

～全卷完～