FINAL FOR CALCULUS

Date: Tuesday, Jan 9, 2003
Instructor: Shu-Yen Pan
No credit will be given for an answer without reasoning.

1. [10%]
(i) Suppose that $xp^3 = 72$. Find $\frac{dp}{dx}$ at $(x, p) = (9, 2)$.
(ii) Check the formula [1] of the integration table below by differentiate the right hand side of the formula.

2. [10%]
(i) Find the limit: $\lim_{x \to -3} \frac{x^2+2x-3}{x^2+5x+6}$.
(ii) Let $f(x) = x^x$. Find $f'(x)$. (Hint: $x = e^{\ln x}$)

3. [10%]
(i) $\int \frac{\ln x}{x\sqrt{1+\ln x}}\, dx$
(ii) $\int \frac{x^3}{\sqrt{x^2+1}}\, dx$.

4. [10%]
(i) Find all the asymptotes of the graph of the function $f(x) = \frac{e^x}{e^x-1}$.
(ii) $\int_{-\infty}^{\infty} \frac{e^{-x}}{(1+e^{-x})^2}\, dx$.

5. [10%]
(i) The Lorenz curve for the distribution of income of country A is $f(x) = \frac{14}{15}x^2 + \frac{1}{15}x$, and the Lorenz curve for country B is $g(x) = \frac{5}{8}x^4 + \frac{3}{8}x$. In which country are the incomes more evenly distributed?
(ii) A company's marginal cost function is $\frac{1}{\sqrt{2x+8}}$ and the fixed costs are 100. Find the cost function.

6. [10%]
(i) Find the derivative $\frac{df}{dx}$ when $f(x) = e^{2x}(x^2 - 1)\ln(x + 1)$.
(ii) An automobile dealer is selling cars at a price of $\$12,000$. The demand function is $D(p) = 2(15 - 0.001p)^2$, where p is the price of a car. Should the dealer raise or lower the price to increase revenue?

7. [10%]
(i) Find the size of the permanent endowment needed to generate an annual $\$2,000$ forever at 10% (annual) interest compounded continuously.
(ii) If the concentration of a drug in the bloodstream after t hours will be

$$A(t) = \frac{c}{b-a}(e^{-at} - e^{-bt})$$

If the constants are $a = 0.4$, $b = 0.6$, and $c = 0.1$, find the time of maximum concentration.

8. [10%] You wish to construct a closed rectangular box that has a volume of 4 cubic meters. The length of the base of the box will be twice as long as its width. The material for the top and bottom of the box costs \$30 per square meter. The material for the sides of the box costs \$20 per square meter. Find the dimensions of the least expensive box that can be constructed.

9. [10%] An oil spill off the coastline was caused by a ruptured tank in a grounded oil tanker. From the aerial photographs, the Coast Guard was able to obtain the dimensions of the oil spill (see picture below). Using Simpson's rule to estimate the area of the oil spill.

10. [10%] Find the area of the region completely enclosed by the graphs of the functions $f(x) = x^3 - 3x + 3$ and $g(x) = x + 3$.

Appendix: Integration Table.

[1] $\int \frac{u}{a+bu}\,du = \frac{1}{b^2}\left[a + bu - a\ln|a+bu|\right] + C$

[2] $\int \frac{u^2}{a+bu}\,du = \frac{1}{2b^3}\left[(a+bu)^2 - 4a(a+bu) + 2a^2\ln|a+bu|\right] + C$

[3] $\int \frac{u}{(a+bu)^2}\,du = \frac{1}{b^2}\left[\frac{a}{a+bu} + \ln|a+bu|\right] + C$

[4] $\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu - 2a)(a+bu)^{3/2} + C$

[5] $\int \frac{u}{\sqrt{a+bu}}\,du = \frac{2}{3b^2}(bu - 2a)\sqrt{a+bu} + C$

[6] $\int \frac{1}{u\sqrt{a+bu}}\,du = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C$

[7] $\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{a}\ln\left|u + \sqrt{a^2+u^2}\right| + C$

[8] $\int u^2\sqrt{a^2+u^2}\,du = \frac{u}{8}(a^2+2u^2)\sqrt{a^2+u^2} - \frac{a^4}{8}\ln\left|u + \sqrt{a^2+u^2}\right| + C$

[9] $\int \frac{1}{\sqrt{a^2+u^2}}\,du = \ln\left|u + \sqrt{a^2+u^2}\right| + C$

[10] $\int \frac{1}{u\sqrt{a^2+u^2}}\,du = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C$

[11] $\int \frac{1}{u^2\sqrt{a^2+u^2}}\,du = -\frac{\sqrt{a^2+u^2}}{a^2u} + C$

[12] $\int \frac{1}{(a^2+u^2)^{3/2}}\,du = \frac{u}{a^2\sqrt{a^2+u^2}} + C$