FINAL FOR CALCULUS

Time: 8:10–10:00 AM, Friday, June 22, 2001
Instructor: Shu-Yen Pan

No calculator is allowed. No credit will be given for an answer without reasoning.

1. [10%] Let $\mathbf{F}(x,y) = x^3y^4\mathbf{i} + x^4y^3\mathbf{j}$. Find a function f such that $\nabla f = \mathbf{F}$ and compute the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the curve $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (1+t^3)\mathbf{j}$, $0 \le t \le 1$.

2. [10%] Use Stoke's theorem to compute the integral $\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS$ where $\mathbf{F}(x,y,z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and S the part of the sphere $x^2+y^2+z^2=4$ that lies inside the cylinder $x^2+y^2=1$ and above the xy-plane.

3. [10%] Find the flux of the vector field $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + y^2\mathbf{k}$ over the unit sphere $x^2+y^2+z^2=1$.

4. [10%] Use Green's theorem to find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

for some $a, b > 0$.

5. [10%] Sketch the solid whose volume is given by the iterated integral

$$\int_0^2 \int_0^{2-x} \int_0^{4-x^2} dz\,dy\,dx.$$

6. [10%] Find local maximum and minimum values and saddle points of the function

$$f(x,y) = x^2 + y^2 + x^2y + 4.$$

7.

(i) Use implicit differentiation to find $\partial z/\partial x$ for $xy^2 + yz^2 + zx^2 = 3$.

(ii) Find a unit normal vector of the plane passing through the point $(0,0,1)$ and spanned by the two vectors $\mathbf{i}+\mathbf{j}$ and $\mathbf{j}-2\mathbf{k}$.

8. [10%] Find the length of the curve $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j} + \ln t\,\mathbf{k}$ for $1 \le t \le e$.

9. [10%] Test the series

$$\sum_{n=1}^{\infty} \frac{\cos n\pi}{\sqrt{n}}$$

for absolutely convergence, conditionally convergence or divergence.

10. [10%] Let $f(x) = \frac{e^x-1}{x}$.

(1) Find the power series representation of f in powers of x.

(2) Differentiate the power series in (1) and show that

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1.$$