FINAL FOR CALCULUS

Time: 8:10–10:00 AM, Friday, January 12, 2000
Instructor: Shu-Yen Pan

No calculator is allowed. No credit will be given for an answer without reasoning.

1. (1) [4%] Find y' for $y = \sqrt{x + \sqrt{x}}$.

(2) [4%] Is $\frac{d}{dx}|x^2 + x| = |2x + 1|$? Why or why not?

2. (1) [4%] Evaluate $\int e^{x+e^x}\, dx$.

(2) [4%] Evaluate $\int_0^1 \ln x\, dx$.

3. (1) [4%] Differentiating the equation $\tan y = x$ implicitly to find $\frac{d}{dx}(\tan^{-1} x)$.

(2) [4%] One model for the spread of a rumor is that the rate of the spread is proportional to the product of the fraction y of the population who have heard the rumor and the faction who have not heard the rumor. Write a differential equation that is satisfied by y.

4. A spinner from a board game randomly indicates a real number between 0 and 10. The spinner is fair in the sense that it indicates a number in a given interval with the same probability as it indicates a number in any other interval of the same length.

(1) [4%] Explain why the function

$$f(x) = \begin{cases} 0.1, & \text{if } 0 \le x \le 10; \\ 0, & \text{if } x < 0 \text{ or } x > 10 \end{cases}$$

is a probability density function for the spinner's values.

(2) [4%] What does your intuition tell you about the value of the mean? Check your answer by evaluating an integral.

5. [6%] Find the arc length function for the curve $y = 2x^{3/2}$ with starting point $P_0(1, 2)$.

6. [6%] If $\lim_{x\to 1}\big(f(x) + g(x)\big) = 2$ and $\lim_{x\to 1}\big(f(x) - g(x)\big) = 6$, find $\lim_{x\to 1} f(x)g(x)$.

7. [8%] If f is a positive function and $f''(x) > 0$ for $a \le x \le b$, show that

$$M_n \le \int_a^b f(x)\, dx \le T_n$$

where M_n is the approximation by midpoint rule and T_n is the approximation by trapezoidal rule.

8. [8%] Find A and B given that the function $y = Ax^{-1/2} + Bx^{1/2}$ has a minimum value 6 at $x = 9$.

9. [8%] Let f be a one-to-one function and $f''(x)$ exists for all x. Let $g = f^{-1}$. Show that

$$g''(x) = -\frac{f''(g(x))}{(f'(g(x)))^3}.$$

10. [8%] Show that the area of a sphere of radius r is $4\pi r^2$.

11. [8%] Find all functions f that satisfy the equation

$$\left(\int f(x)\, dx\right)\left(\int \frac{1}{f(x)}\, dx\right) = -4.$$

12. [8%] A student forgot the product rule for differentiation and made the mistake of thinking that $(fg)' = f'g'$. However, she was lucky and got the correct answer. The function f that she used was $f(x) = e^{x^2}$ and the domain of her problem was the interval $(\frac{1}{2}, \infty)$. What was the function g?

13. [8%] Evaluate $\lim_{x\to 2}\big(\frac{x}{x-2}\int_2^x e^{t^2}\, dt\big)$.