MIDTERM 1 FOR CALCULUS

Date: 2000, June 1, 8:10–10:00AM
Each problem is worth 10 points.

1. Suppose that $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = -2\mathbf{i} + 3\mathbf{k}$. Find:

(i) $\mathbf{a} \cdot \mathbf{b}$,
(ii) $\mathbf{b} \times 2\mathbf{a}$,
(iii) $(\mathbf{b} \times \mathbf{a}) \times \mathbf{b}$,
(iv) $\text{proj}_{\mathbf{b}}\mathbf{a}$,
(v) $\|a\|$.

2.

(i) Let l_1, l_2 be lines that pass through the origin and have direction vectors $\mathbf{d}_1 = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{d}_2 = -\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ respectively. Find an equation for the plane that contains l_1 and l_2.
(ii) Show that $4\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a} + \mathbf{b}\|^2 - \|\mathbf{a} - \mathbf{b}\|^2$.

3. Find the limits:

(i) $\lim_{n\to\infty} \frac{\sin n}{n}$,
(ii) $\lim_{x\to e} \frac{\ln(\ln x)}{\ln x - 1}$,
(iii) $\lim_{x\to 0}(\cosh x)^{4/x}$
(iv) $\lim_{x\to 0^+} \sin x \ln x$.

4.

(i Determine whether $\sum_{k=1}^{\infty} \left(\frac{\ln k}{k}\right)^k$ converges or diverges.
(ii) Determine whether $\sum_{k=1}^{\infty} (-1)^k \frac{2\sqrt{k}}{k^2+1}$ converges absolutely, converges conditionally or diverges.

5. A nonnegative function f defined on $(-\infty, \infty)$ is a *probability density function* if

$$\int_{-\infty}^{\infty} f(x)\, dx = 1.$$

(i) Let $k > 0$. Show that the function f defined by

$$f(x) = \begin{cases} ke^{-kx}, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0 \end{cases}$$

is a probability density function.
(ii) Let f be defined as in (i). Compute the *mean* μ given by

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx.$$

6. Find the interval of convergence for

$$\sum_{k=1}^{\infty} \frac{k^2 + k}{x^k}.$$

7. Evaluate

(i) $\int_0^3 \frac{x}{(x^2-1)^{3/2}}\, dx$,
(ii) $\int_{-\infty}^{1} e^{(x - e^x)}\, dx$.

8. Let $f(x) = \frac{e^x - 1}{x}$.

(i) Find the power series representation of f in powers of x.

(ii) Differentiate the power series in (i) and show that

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1.$$

9. Suppose that the function $f(x)$ is infinitely differentiable on an interval containing 0, and suppose that $f''(x) + f(x) = 1$ and $f(0) = 1,\ f'(0) = 1$. Find the power series representation of f in powers of x. Find the radius of convergence.

10. Let a and b be positive numbers with $b > a$. Define two sequences $\{a_n\}$ and $\{b_n\}$ as follows:

$$\begin{aligned} a_1 &= \tfrac{a+b}{2} \\ b_1 &= \sqrt{ab} \\ a_n &= \tfrac{a_{n-1}+b_{n-1}}{2} \\ b_n &= \sqrt{a_{n-1}b_{n-1}}, \qquad \text{for } n = 2, 3, 4, \ldots \end{aligned}$$

(i) Use mathematical induction to show that $a_{n-1} > a_n > b_n > b_{n-1}$ for $n = 2, 3, 4, \ldots$.

(ii) Prove that $\{a_n\}$ and $\{b_n\}$ are convergent sequences and that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.